Boundary value problems for complex partial differential equations in plane domains

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There are two principally different complex partial differential operators of one complex variable, the pollyanalytic and the polyharmonic operators, $\partial_{\overline{z}}^n$ and $(\partial_z\partial_{\overline{z}})^n$, $n \in \mathbb{N}$. The general complex differential operator $\partial_z^m \partial_{\overline{z}}^n, m, n \in \mathbb{N}$, can obviously be decomposed in the product of a polyanalytic (or anti-polyanalytic $\partial_z^k . k \in \mathbb{N}$) and a polyharmonic operator. Using their potentials general linear differential operators with them as leading term can be treated, see [7,1]. Basic boundary value problems for the simplest operators are in the case of the Cauchy-Riemann operator $\partial_{\overline{z}}$ the Schwarz, Dirichlet, Neumann, Robin and more generally the Riemann and Riemann-Hilbert problems and in case of the Laplace operator $\partial_z\partial_{\overline{z}}$ the Robin problem with its particular cases, the Dirichlet and Neumann problems. While the Schwarz problem for the Cauchy-Riemann operator is well-posed the others are not [2]. The Robin problem for the Poisson equation $\partial_z\partial_{\overline{z}}w = f$ in general is only conditionally solvable. Only the Dirichlet problem is unconditionally solvable. The solution to the Robin problem is explicitly expressed through the Robin function, a particularly adjusted fundamental solution to the Laplacian with the Green and the Neumann functions as special cases. While the existence of Green functions even for general elliptic operators is well studied, explicit expressions for them are rarely investigated although they would provide explicit solutions to boundary value problems important in applications. There are three methods to calculate the harmonic Green function for plane domains based on i. its conformal invariance, ii. the Schwarz problem for analytic functions, and iii. the parqueting-reflection method. Green functions are e.g. known for disc sectors, cones, certain convex polygons like an equilateral triangle [4], half hexagon, half rings, certain lunes and lenses [5], rings. The Schwarz problem for the inhomogeneous polyanalytic, for $n = 2$ Bitsadze, equation is explicitly solved for the unit disc [3]. For the $n-$Poisson equation $(\partial_z\partial_{\overline{z}})^n w = f$ there exists a variety of boundary value problems [6], the larger n is the more conditions are possible. Besides the Dirichlet and Neumann problems there is e.g. the Requir (Navier) problem. The latter problem is decomposable in a set of coupled Dirichlet problems for the Poisson equation. Incorporating also Neumann and Robin boundary conditions this leads to a hierarchy of hybrid polyharmonic Green functions.

References

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