Théorie des groupes/Group Theory
Théorie des Nombres/Number Theory

A FORMAL THEORY OF EISENSTEIN SERIES

Kâzim İlhan İkeda

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Abstract. The definition of the Eisenstein series $\boldsymbol{E}(Z,s;k,\Gamma)$ on the symplectic and unitary groups G is well-known. We introduce a series $\mathcal{E}(Z,\lambda)$ whose construction involves a 5-tuple $\{G,P,M,W,\epsilon\}$ satisfying certain conditions. We prove that, if $\lambda : M_{n\times 2n}(\mathfrak{k}) \longrightarrow \mathbb{C}$ is a locally-constant function, then the series $\mathcal{E}(Z,\lambda)$ is a finite linear combination of G-transforms of the Eisenstein series $\boldsymbol{E}(Z,s;k,\Gamma)$ whose coefficients are products of certain Hecke L-functions.

UNE THÉORIE FORMELLE DES SÉRIES d'EISENSTEIN

Résumé. Sur les groupes symplectique et unitaire G la définition des séries d'Eisenstein $\boldsymbol{E}(Z,s;k,\Gamma)$ est bien connue. Dans cette note nous introduisons une série $\mathcal{E}(Z,\lambda)$ dont la construction fait intervenir un 5-tuplet $\{G,P,M,W,\epsilon\}$ satisfaisant à certaines conditions. Nous démontrons que, si $\lambda : M_{n\times 2n}(\mathfrak{k}) \longrightarrow \mathbb{C}$ est une fonction localement constante, alors la série $\mathcal{E}(Z,\lambda)$ est une combinaison linéaire finie des G-transformés des séries d'Eisenstein $\boldsymbol{E}(Z,s;k,\Gamma)$ dont les coefficients sont produits de certaines fonctions L de Hecke.

Version française abrégée. §1. Considérons la donnée (que nous appelons une *donnée d'Eisenstein* dans la suite) définie par un 5-tuplet $\{G,P,M,W,\epsilon\}$ où $\epsilon \in W$ est un ensemble non-vide, G et M sont des groupes de transformations sur W, et P est un sous-groupe de G satisfaisant aux conditions suivantes:
(i) *G opère transitivement à droite sur W et M opére fidèlement à gauche de W. Ces actions sur W sont compatibles comme dans (1);* **(ii)** *$T_\epsilon \subset P$ où T_ϵ désigne le groupe d'isotropie de ϵ dans G;* **(iii)** *Il existe un morphisme surjectif $\underline{d} : P \longrightarrow M$ tel que le carré (2) est commutatif, où r_ϵ est défini par l'action à droite de P sur ϵ et l_ϵ est l'application définie par l'action à gauche de M sur ϵ;* **(iv)** *Il existe un G-espace $\mathfrak{H}$ et un groupe $\mathcal{G} = GL_*(\mathbb{C})$ avec un facteur d'automorphie $j : G \times \mathfrak{H} \longrightarrow \mathcal{G}$ à valeurs dans $\mathcal{G}$ tel que la condition (3) est satisfaite pour tous $\pi \in P$ et $z \in \mathfrak{H}$; où $\delta : M \longrightarrow \mathcal{G}$ est un morphisme.*
§2. (*Tous les calculs du §2 sont formels.*) Soit Γ un sous-groupe de G et soit Δ un sous-groupe de M vérifiant la condition (4). Etant donné un caractère $\psi : \Gamma \to \mathbb{T}$, vérifiant (7), où B désigne un système complet de représentants de $P\backslash G/\Gamma$, on considère les séries $\mathcal{E}(z,\lambda,\psi)$ comme dans (8), où $\lambda : W \to \mathbb{C}$ est une application quelconque vérifiant (6). La série d'Eisenstein $\boldsymbol{E}(z,\Gamma,\psi)$ de G (relative à Γ et à ψ) est définie comme dans (9). On démontre au §2 (Théorème 2.3) que la série $\mathcal{E}(z,\lambda,\psi)$ est une combinaison linéaire des β-transformés des séries d'Eisenstein $\boldsymbol{E}(z,\beta\Gamma\beta^{-1},\psi_\beta)$ pour $\beta \in B$.

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§3. Dans cette section, F désigne un corps de nombres totalement réel de degré m et K désigne un CM-corps ayant F pour sous-corps totalement réel maximal. Pour uniformiser notre exposé, le symbole $\mathfrak{k}$ désigne soit F, soit K s'il n'y a aucune confusion et $O_{\mathfrak{k}}$ désigne l'anneau des entiers de $\mathfrak{k}$. On note $\boldsymbol{a}$ l'ensemble $\boldsymbol{a}_F$ des premiers archimédiens de F ou bien d'un CM-type fixé de K, et $\boldsymbol{h}$ désigne l'ensemble des premiers non-archimédiens de F. Pour $X \in M_{n\times s}(K)$, nous posons $X^* = {}^tX^{\iota}$, où $\iota : K \longrightarrow K$ est l'involution de Galois sur F, et $J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{Q})$. Nous considérons les groupes algébriques $Sp(n, F)$ et $SU(n, K)$ définis sur F que nous noterons simplement par $G_{\mathfrak{k}}$. Nous introduisons $\widetilde{G}_{\mathfrak{k}}$ comme dans (12). Notons $P_{\mathfrak{k}}$ le sous-groupe parabolique de $G_{\mathfrak{k}}$ correspondant à la partition (n, n), $Q_{\mathfrak{k}}$ la composante de Levi et $R_{\mathfrak{k}}$ le radical unipotent de $P_{\mathfrak{k}}$. Les sous-groupes $\widetilde{P}_{\mathfrak{k}}$, $\widetilde{Q}_{\mathfrak{k}}$ et $\widetilde{R}_{\mathfrak{k}}$ de $\widetilde{G}_{\mathfrak{k}}$ sont définis de la même manière. Soit $W_{\mathfrak{k}} = \{\alpha \in M_{n\times 2n}(\mathfrak{k}) \mid \alpha J_n \alpha^* = 0,\ \mathrm{rang}(\alpha) = n\}$. On démontre que $\{\widetilde{G}_{\mathfrak{k}}, \widetilde{P}_{\mathfrak{k}}, GL_n(\mathfrak{k}), W_{\mathfrak{k}}, (0_n, 1_n)\}$ est une donnée d'Eisenstein. Pour un sous-groupe congruence Γ de $\widetilde{G}_{\mathfrak{k}}$ (cf **[6]** pp 424), définissons la série d'Eisenstein $\boldsymbol{E}(Z, s; k, \Gamma)$ comme dans (15), en supposant que $\prod_{\nu\in\boldsymbol{a}} \det(\mu_\nu(x_\nu, z_\nu))^k = 1$ pour tout $x \in P_{\mathfrak{k}} \cap \Gamma$ pour que la somme ait un sens. Soit Δ un sous-groupe de congruence de $GL_n(\mathfrak{k})$ suffisamment petit pour que $\delta(\Delta) = 1$. Notons par $\mathcal{S}(V)$ l'espace des fonctions localement constantes définies sur un espace vectoriel V sur $\mathbb{Q}$ à valeurs dans $\mathbb{C}$. Considérons un élément $\lambda \in \mathcal{S}(M_{n\times 2n}(\mathfrak{k}))$, tel que $\lambda(\Delta w \Gamma) = \lambda(w)$ pour $w \in M_{n\times 2n}(\mathfrak{k})$. Suivant la notation de (5), introduisons la somme $\mathcal{E}(Z, \lambda)$ comme dans (16).

Théorème 3.1. *Soit Γ un sous-groupe de congruence de $\widetilde{G}_{\mathfrak{k}}$ satisfaisant à la condition $\prod_{\nu\in\boldsymbol{a}} det(\mu_\nu(x_\nu, z_\nu))^k = 1$ pour tout $x \in P_{\mathfrak{k}} \cap \Gamma$ et tout $Z = (z_\nu)_{\nu\in\boldsymbol{a}} \in \mathfrak{H}^{\boldsymbol{a}}$ et soit Δ un sous-groupe de congruence de $GL_n(\mathfrak{k})$ tel que $\delta(\Delta) = 1$. Soit $\lambda : M_{n\times 2n}(\mathfrak{k}) \longrightarrow \mathbb{C}$ une fonction localement constante avec la propriété $\lambda(\Delta w \Gamma) = \lambda(w)$ pour $w \in M_{n\times 2n}(\mathfrak{k})$. Alors, la série $\mathcal{E}(Z, \lambda)$ est une combinaison linéaire finie des β-transformés $\boldsymbol{E}(Z, s : k, \beta\Gamma\beta^{-1})$ de la série d'Eisenstein dont les coefficients sont des sommes de produits de séries L de Hecke.*

1.PRELIMINARIES. Consider the data (which we call an *Eisenstein datum* in the sequel) defined by a 5-tuple $\{G, P, M, W, \epsilon\}$, where $\epsilon \in W$ is a (non-empty) set, G and M transformation groups on W and P a subgroup of G satisfying the following conditions:

(i) G acts transitively on the right of W and M acts faithfully on the left of W. These actions on W are compatible in the following sense:

(1) $$(mw)g = m(wg), \text{ for every } m \in M,\ w \in W \text{ and } g \in G.$$

(ii) $T_\epsilon \subset P$, where T_ϵ denote the isotropy group of ϵ in G.
(iii) There exist a surjective morphism $\underline{d} : P \longrightarrow M$ such that the square

(2) $$\begin{array}{ccc} P & \xrightarrow{r_\epsilon} & W \\ {\scriptstyle \mathrm{id}_P}\downarrow & & \uparrow{\scriptstyle l_\epsilon} \\ P & \xrightarrow{\underline{d}} & M \end{array}$$

is commutative, where $r_\epsilon : P \longrightarrow W$ *is defined via the right action of* P *on* ϵ *and* $l_\epsilon : M \longrightarrow W$ *is the map defined by the left action of* M *on* ϵ.
(iv) There exists a G*-space* $\mathfrak{H}$ *and a group* $\mathcal{G} = GL_*(\mathbb{C})$ *with a* $\mathcal{G}$*-valued factor of automorphy* $j : G \times \mathfrak{H} \longrightarrow \mathcal{G}$ *(that is the cocycle relation* $j(g_1g_2, z) = j(g_1, g_2(z))j(g_2, z)$*, for* $g_1, g_2 \in G$ *and for* $z \in \mathfrak{H}$ *is satisfied) such that*

$$j(\pi, z) = \delta(\underline{d}(\pi)), \tag{3}$$

for every $\pi \in P$ *and* $z \in \mathfrak{H}$*; where* $\delta : M \longrightarrow \mathcal{G}$ *is a morphism.*

Let $\{G, P, M, W, \epsilon\}$ be an Eisenstein datum. Define the mapping $\tau : G \to W$ by $g \mapsto \tau(g) = \epsilon g$, for $g \in G$. Then we readily have the following

Lemma 1.1. **(i)** *The map* $\tau : G \longrightarrow W$ *is a surjection with* $\tau(1) = \epsilon$, $\tau(g_1g_2) = \tau(g_1)g_2$ *and* $\tau(\pi g) = \underline{d}(\pi)\tau(g)$ *for every* $g, g_1, g_2 \in G$ *and* $\pi \in P$*;* **(ii)** $T_\epsilon = Ker(\underline{d})$*;* **(iii)** $j(g, z) = 1$ *for every* $g \in T_\epsilon$ *and* $z \in \mathfrak{H}$*;* **(iv)** *The mapping* $\tau : G \longrightarrow W$ *induces a bijection* $\iota_\tau : T_\epsilon \backslash G \longrightarrow W$ *defined by* $T_\epsilon g \longmapsto \tau(g)$ *for* $g \in G$.

2.A FORMAL THEORY OF EISENSTEIN SERIES. *In this paragraph all of the computations are formal and we ignore the convergence questions.* Let Γ be a subgroup of G and B be a complete set of representatives of the double-coset decomposition $P\backslash G/\Gamma$ of G with respect to (P, Γ). That is, there is the disjoint union $G = \bigsqcup_{\beta \in B} P\beta\Gamma$. For $\beta \in B$, $P \cap \beta\Gamma\beta^{-1}$ becomes a transformation group on the set $\beta\Gamma$, where the action of $P \cap \beta\Gamma\beta^{-1}$ on $\beta\Gamma$ is defined via left multiplication. Let S_β be a complete set of representatives of the orbits in $\beta\Gamma$ relative to $P \cap \beta\Gamma\beta^{-1}$. The proof of the following Lemma is elementary, so it is omitted.

Lemma 2.1. **(i)** $G = \bigsqcup_{\beta \in B} PS_\beta$*;* **(ii)** *The set* W *is the disjoint union of the sets* $M\epsilon S_\beta$ *and the map defined by* $(m, x) \mapsto m\epsilon x$ *is a bijection from* $\bigsqcup_{\beta \in B} M \times S_\beta$ *to* W.

Let Δ be a subgroup of M (then Δ is a transformation group on W, where the action of Δ on W is induced from that of M on W). The orbits $\Delta\backslash W$ in W relative to Δ have the following description:

Lemma 2.2. $\Delta\backslash W = \bigsqcup_{\beta \in B}\{r\epsilon x \mid r \in \Delta\backslash M, x \in S_\beta\}$.

Assume that

$$\underline{d}(P \cap \beta\Gamma\beta^{-1}) \subset \Delta \subset \mathrm{Ker}(\delta), \tag{4}$$

for every $\beta \in B$. So there is the canonical induced *mapping* $\delta_* : \Delta\backslash M \longrightarrow \mathcal{G}$. As a notation (which are clearly well-defined) introduce:

$$j(g, z) = j(\tau(g), z) \text{ and } j(x, z) = j(w, z), \tag{5}$$

for $g \in G$, $x = \Delta w \in \Delta\backslash W$ and $z \in \mathfrak{H}$. Take $\lambda : W \longrightarrow \mathbb{C}$ any map satisfying the condition

$$\lambda(\Delta w\gamma) = \psi(\gamma)\lambda(w), \tag{6}$$

for any $\gamma \in \Gamma$ and $w \in W$ with a character $\psi : \Gamma \longrightarrow \mathbb{T} = \{z \in \mathbb{C} :\mid z \mid= 1\}$ such that

$$P \cap \beta\Gamma\beta^{-1} \subset \mathrm{Ker}(\psi), \tag{7}$$

for every $\beta \in B$. We now introduce the following series:

$$(8) \qquad \mathcal{E}(.,\lambda,\psi):\mathfrak{H} \longrightarrow \mathcal{G} \text{ defined by } \mathcal{E}(z,\lambda,\psi)=\sum_{x\in\Delta\backslash W}\lambda(x)j(x,z),$$

for $z \in \mathfrak{H}$, which is a well-defined function by virtue of the assumption (6).

$$(9) \qquad \boldsymbol{E}(.,\Gamma,\psi):\mathfrak{H} \longrightarrow \mathcal{G} \text{ defined by } \boldsymbol{E}(z,\Gamma,\psi)=\sum_{x\in(P\cap\Gamma)\backslash\Gamma}\psi(x)j(x,z),$$

for $z \in \mathfrak{H}$, which is a well-defined function, since $\underline{d}(P\cap\Gamma) \subset \Delta \subset \mathrm{Ker}(\delta)$ and $P\cap\Gamma \subset \mathrm{Ker}(\psi)$. The series introduced in (9) is the Eisenstein series of G (with respect to Γ and $\psi:\Gamma \longrightarrow \mathbb{T}$). For $\beta \in B$, denote by $\psi_\beta : \beta\Gamma\beta^{-1} \longrightarrow \mathbb{T}$ to be the character which factors through

$$(10) \qquad \psi_\beta : \beta\Gamma\beta^{-1} \xrightarrow{\beta^{-1}-\text{conjugation}} \Gamma \xrightarrow{\psi} \mathbb{T}.$$

Consider the Eisenstein series $\boldsymbol{E}(z,\beta\Gamma\beta^{-1},\psi_\beta)$, which is well-defined since $\underline{d}(P\cap\beta\Gamma\beta^{-1}) \subset \Delta \subset \mathrm{Ker}(\delta)$ and $P\cap\beta\Gamma\beta^{-1} \subset \mathrm{Ker}(\psi)$. Then for $\beta \in B$, the β-transform of $\boldsymbol{E}(.,\beta\Gamma\beta^{-1},\psi_\beta):\mathfrak{H} \longrightarrow \mathcal{G}$ is given by

$$(11) \qquad \boldsymbol{E}||_\beta(z,\beta\Gamma\beta^{-1},\psi_\beta)=\sum_{x\in S_\beta}\psi(\beta^{-1}x)j(x,z), \text{ for } z\in\mathfrak{H}.$$

We now state the main theorem: the series $\mathcal{E}(z,\lambda,\psi)$ is a linear combination of β-transforms of Eisenstein series $\boldsymbol{E}(z,\beta\Gamma\beta^{-1},\psi)$ for $\beta \in B$. More precisely,

Theorem 2.3. *For any $z \in \mathfrak{H}$, $\mathcal{E}(z,\lambda,\psi) = \sum_{\beta\in B}\mathcal{L}_\beta \boldsymbol{E}||_\beta(z,\beta\Gamma\beta^{-1},\psi_\beta)$ where $\mathcal{L}_\beta = \sum_{r\in\Delta\backslash M}\lambda(r\epsilon\beta)\delta_*(r)$ for $\beta \in B$.*

3. APPLICATION OF THE FORMAL THEORY OF EISENSTEIN SERIES. Let F be a totally real algebraic number field of degree m and K a CM-field with the maximal totally real subfield F (i.e K is a totally imaginary quadratic extension of F). To make our exposition uniform, the symbol $\mathfrak{k}$ will denote either F or K if there is no fear of confusion and $O_\mathfrak{k}$ will denote the ring of integers of $\mathfrak{k}$. We denote by $\boldsymbol{a}$ either $\boldsymbol{a}_F$ the set of archimedean primes of F or a fixed CM-type of K and $\boldsymbol{h}$ will denote the set of non-archimedean primes of F. We call ι the Galois involution of K/F. Let $\widetilde{G}_\mathfrak{k}$ be the algebraic group defined over F by

$$(12) \qquad \widetilde{G}_\mathfrak{k} = \{X \in GL_{2n}(\mathfrak{k}) \mid XJ_nX^* = J_n\},$$

where $J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{Q})$ and X^* denotes respectively, the transpose ${}^tX$ in the totally real case $\mathfrak{k} = F$, and the transpose-conjugate ${}^tX^\iota$ in the CM-case $\mathfrak{k} = K$. We denote the group $\widetilde{G}_\mathfrak{k} \cap SL_{2n}(\mathfrak{k})$ simply by $G_\mathfrak{k}$. Let P be the parabolic subgroup of G corresponding to the partition (n,n), Q the Levi component and R the unipotent radical of P. The subgroups $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{R}$ of $\widetilde{G}$ are defined likewise. Let $W_\mathfrak{k} = \{\alpha \in M_{n\times 2n}(\mathfrak{k}) \mid \alpha J_n\alpha^* = 0, \mathrm{rank}(\alpha) = n\}$. The surjection $\tau : \widetilde{G}_\mathfrak{k} \longrightarrow W_\mathfrak{k}$ defined by

$\tau(X) = (0_n, 1_n)X = (c_X, d_X)$ for $X = \begin{pmatrix} a_X & b_X \\ c_X & d_X \end{pmatrix} \in \widetilde{G}_{\mathfrak{k}}$ induces a right transitive action of $\widetilde{G}_{\mathfrak{k}}$ on $W_{\mathfrak{k}}$ as $\tau(X)Y = \tau(XY)$ for X and Y elements of $\widetilde{G}_{\mathfrak{k}}$. Let $\underline{d} : \widetilde{P}_{\mathfrak{k}} \longrightarrow GL_n(\mathfrak{k})$ be the surjective homomorphism defined as $\underline{d}(X) = d_X$ for $X = \begin{pmatrix} a_X & b_X \\ 0 & d_X \end{pmatrix} \in \widetilde{P}_{\mathfrak{k}}$. Then we have the commutative square (2). The group $GL_n(\mathfrak{k})$ acts faithfully on $W_{\mathfrak{k}}$ from the left by $m(c, d) = (mc, md)$ for $m \in GL_n(\mathfrak{k})$ and $(c, d) \in W_{\mathfrak{k}}$. The right action of $\widetilde{G}_{\mathfrak{k}}$ and the left action of $GL_n(\mathfrak{k})$ on $W_{\mathfrak{k}}$ are clearly compatible. The isotropy subgroup $T_{(0_n,1_n)}$ of $(0_n, 1_n)$ in $\widetilde{G}_{\mathfrak{k}}$ is the unipotent radical $\widetilde{R}_{\mathfrak{k}}$ of $\widetilde{P}_{\mathfrak{k}}$. The local archimedean group $\widetilde{G}_{\mathfrak{k},\nu}$ ($\nu \in \boldsymbol{a}$) acts on the symmetric space

$$\mathfrak{H}_{\mathfrak{k}} = \begin{cases} \{Z \in M_n(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}, & \mathfrak{k} = F, \\ \{Z \in M_n(\mathbb{C}) \mid i({}^t\overline{Z} - Z) > 0\}, & \mathfrak{k} = K, \end{cases}$$

as $\alpha(Z) = (a_\alpha Z + b_\alpha)(c_\alpha Z + d_\alpha)^{-1}$ for $\alpha \in \widetilde{G}_{\mathfrak{k},\nu}$ and $Z \in \mathfrak{H}_{\mathfrak{k}}$. The map $\mu_\nu : \widetilde{G}_{\mathfrak{k},\nu} \times \mathfrak{H}_{\mathfrak{k}} \longrightarrow GL_n(\mathbb{C})$ defined by $\mu_\nu(\alpha, z) = c_\alpha z + d_\alpha$ is a $GL_n(\mathbb{C})$-valued factor of automorphy of $\widetilde{G}_{\mathfrak{k},\nu}$ on $\mathfrak{H}_{\mathfrak{k}}$. Following the notation of **[6]**, we introduce

$$J_{k,s}(x, Z) = \prod_{\nu \in \boldsymbol{a}} \det(\mu_\nu(x_\nu, z_\nu))^k \mid \det(\mu_\nu(x_\nu, z_\nu)) \mid^s, \tag{13}$$

where $Z = (z_\nu)_{\nu \in \boldsymbol{a}} \in \mathfrak{H}_{\mathfrak{k}}^{\boldsymbol{a}}$, $x \in G \hookrightarrow G_{\mathbb{A}}$, $k \in \mathbb{Z}$ and $s \in \mathbb{C}$. It is clear that

$$J_{k,s}(\pi, Z) = \mathrm{N}_{\mathfrak{k}/\mathbb{Q}}(\det(\underline{d}(\pi)))^{k/[\mathfrak{k}:F]} \mid \mathrm{N}_{\mathfrak{k}/\mathbb{Q}}(\det(\underline{d}(\pi))) \mid^{s/[\mathfrak{k}:F]}, \text{ for } \pi \in \widetilde{P}_{\mathfrak{k}}. \tag{14}$$

So the morphism $\delta : GL_n(\mathfrak{k}) \longrightarrow \mathbb{C}^\times$, $\delta(g) = \mathrm{N}_{\mathfrak{k}/\mathbb{Q}}(\det(g))^{k/[\mathfrak{k}:F]} \mid \mathrm{N}_{\mathfrak{k}/\mathbb{Q}}(\det(g)) \mid^{s/[\mathfrak{k}:F]}$ for $g \in GL_n(\mathfrak{k})$ satisfies $\delta(\underline{d}(\pi)) = J_{k,s}(\pi, Z)$ for every $\pi \in \widetilde{P}_{\mathfrak{k}}$ and $Z \in \mathfrak{H}_{\mathfrak{k}}^{\boldsymbol{a}}$. Hence we have shown that: $\{\widetilde{G}_{\mathfrak{k}}, \widetilde{P}_{\mathfrak{k}}, GL_n(\mathfrak{k}), W_{\mathfrak{k}}, (0_n, 1_n)\}$ is an Eisenstein datum.

For a congruence subgroup Γ of $\widetilde{G}_{\mathfrak{k}}$ (cf **[6]** pp 424), define the Eisenstein series

$$\boldsymbol{E}(Z, s; k, \Gamma) = \sum_{\alpha \in (P_{\mathfrak{k}} \cap \Gamma)\backslash\Gamma} J_{k,s}(\alpha, Z)^{-1} \tag{15}$$

for $Z = (z_\nu)_{\nu \in \boldsymbol{a}} \in \mathfrak{H}_{\mathfrak{k}}^{\boldsymbol{a}}$, $s \in \mathbb{C}$, $k \in \mathbb{Z}$; provided that $\prod_{\nu \in \boldsymbol{a}} \det(\mu_\nu(x_\nu, z_\nu))^k = 1$ for every $x \in P_{\mathfrak{k}} \cap \Gamma$ to make the sum meaningful. The series $\boldsymbol{E}(Z, s; k, \Gamma)$ converges for $\mathrm{Re}(k + s) > \begin{cases} n + 1 & (\mathfrak{k} = F) \\ 2n & (\mathfrak{k} = K) \end{cases}$. Let Δ be a sufficiently small congruence subgroup of $GL_n(\mathfrak{k})$ such that $\delta(\Delta) = 1$. Let $\mathcal{S}(V)$ denote the space of locally constant functions $V \longrightarrow \mathbb{C}$ on a vector space V over $\mathbb{Q}$. Recall that a function $\ell : V \rightarrow \mathbb{C}$ is called locally constant, if there exist two lattices L_1 and L_2 of V so that $\ell(x) = 0$ if $x \notin L_1$ and $\ell(x) = \ell(y)$ if $x \equiv y \pmod{L_2}$. Consider an element $\lambda \in \mathcal{S}(M_{n\times 2n}(\mathfrak{k}))$, such that $\lambda : M_{n\times 2n}(\mathfrak{k}) \longrightarrow \mathbb{C}$ satisfies the condition $\lambda(\Delta w \Gamma) = \lambda(w)$ for $w \in M_{n\times 2n}(\mathfrak{k})$. Following the notation (5), introduce the sum

$$\mathcal{E}(Z, \lambda) = \sum_{x \in \Delta\backslash W_{\mathfrak{k}}} \lambda(x) J_{k,s}(x, Z)^{-1}, \tag{16}$$

for $Z \in \mathfrak{H}_{\mathfrak{k}}^{\boldsymbol{a}}$. Theorem 2.3 yields $\mathcal{E}(Z, \lambda) = \sum_{\beta \in B} \mathcal{L}_\beta \boldsymbol{E} \parallel_\beta (Z, s; k, \beta\Gamma\beta^{-1})$, where B is a complete set of representatives of $\widetilde{P}_{\mathfrak{k}}\backslash\widetilde{G}_{\mathfrak{k}}/\Gamma$ which is known to be a finite set and $\mathcal{L}_\beta = \sum_{r \in \Delta\backslash GL_n(\mathfrak{k})} \lambda(r\tau(\beta)) \mathrm{N}_{\mathfrak{k}/\mathbb{Q}}(\det(r))^{-k/[\mathfrak{k}:F]} \mid \mathrm{N}_{\mathfrak{k}/\mathbb{Q}}(\det(r)) \mid^{-s/[\mathfrak{k}:F]}$. Such Dirichlet series are studied by Shimura in **[7]** (pp. 309-313). We state the main theorem of this paragraph. The proof utilizes Proposition 9.2 of **[7]**.

Theorem 3.1. *Let Γ be a congruence subgroup of $\widetilde{G}_{\mathfrak{k}}$ satisfying the condition $\prod_{\nu\in\boldsymbol{a}} det(\mu_\nu(x_\nu, z_\nu))^k = 1$ for every $x \in P_{\mathfrak{k}} \cap \Gamma$ and $Z = (z_\nu)_{\nu\in\boldsymbol{a}} \in \mathfrak{H}_{\mathfrak{k}}^{\boldsymbol{a}}$ and Δ be a sufficiently small congruence subgroup of $GL_n(\mathfrak{k})$ such that $\delta(\Delta) = 1$. Let $\lambda : M_{n\times 2n}(\mathfrak{k}) \longrightarrow \mathbb{C}$ be a locally constant function with the property $\lambda(\Delta w \Gamma) = \lambda(w)$ for $w \in M_{n\times 2n}(\mathfrak{k})$. Then, the series $\mathcal{E}(Z, \lambda)$ is a finite linear combination of β-transforms of the Eisenstein series $\boldsymbol{E}(Z, s; k, \beta\Gamma\beta^{-1})$ whose coefficients are sums of products of certain Hecke L-series. More precisely,*

$$\mathcal{E}(Z,\lambda) = \sum_{\beta\in B}\left[\sum_{1\le j\le k(\beta)} a_{\beta j} b_{\beta j}^{\frac{s}{[\mathfrak{k}:F]}} \prod_{0\le i\le n-1} L(\frac{s}{[\mathfrak{k}:F]} - i, \chi_{n-i}\psi_{\beta j i})\right] \boldsymbol{E} \,||_\beta\, (Z, s; k, \beta\Gamma\beta^{-1})$$

where $a_{\beta j} \in \mathbb{C}$, $0 < b_{\beta j} \in \mathbb{Q}$ and $\psi_{\beta j i}$ are Hecke characters of $\mathfrak{k}_{\mathbb{A}}^{\times}$ of finite order for $\beta \in B$, $1 \le j \le k(\beta)$ and $0 \le i \le n-1$.

If $\mathrm{Re}(s)$ is sufficiently large, then all the series introduced in this paragraph are absolutely convergent and the computations make sense. The proof of the fact that Eisenstein series $\boldsymbol{E}(Z, s; k, \Gamma)$ can be recovered from the series of the type $\mathcal{E}(Z, \lambda)$ is much more involved and will be published as a separate paper.

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Division of Arithmetic, Department of Mathematics, Research Institute for Basic Sciences, TÜBİTAK-Marmara Research Center, P.O Box 21, 41470 Gebze, Turkey

E-mail address: `ikeda@yunus.mam.tubitak.gov.tr` *Fax*: +90 262 641 2309 *Phone*:+90 262 641 2300