manuscripta math. 132, 19–49 (2010)

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Non-abelian local reciprocity law

Received: 2 April 2009 / Revised: 12 October 2009
Published online: 4 February 2010

Abstract. Following our work on the generalized Fesenko reciprocity map, we construct the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ of a local field K as a certain isomorphism from the absolute Galois group G_K of K onto a topological group $\nabla_{K,Y}^{(\varphi)}$ whose definition involves Fontaine–Wintenberger theory of field of norms, and build the non-abelian local class field theory over K in the sense of Fesenko and Koch.

1. Introduction

Let K be a local field; that is a complete discrete valuation field with finite residue class field κ_K of $q = p^f$ elements. For technical reasons, all through the text except the last section, we assume that the multiplicative group $\boldsymbol{\mu}_p(K^{sep})$ of all pth roots of unity in K^{sep} further satisfies $\boldsymbol{\mu}_p(K^{sep}) \subset K$. Fix a Lubin–Tate splitting φ over K. That is, we fix an extension φ of the Frobenius automorphism of K^{nr} to K^{sep} (for details, cf. [13]). In this paper, which is the natural continuation of [11] and [12], we construct the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ for K, which is an isomorphism from the absolute Galois group G_K of K onto a certain topological group $\nabla_{K,Y}^{(\varphi)}$ (cf. (6.11) and (6.17) in Sect. 6) and furthermore study the basic functorial properties of $\boldsymbol{\Phi}_K^{(\varphi)}$. Moreover, we sketch the construction of the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ of K, where K is any local field not necessarily satisfying $\boldsymbol{\mu}_p(K^{sep}) \subset K$ via non-abelian Schreier theory in the profinite regime.

The organization of the paper is as follows. In Sect. 2, we briefly review the main results of [12] on the generalized Fesenko reciprocity map, which will play

To the memory of Cahit Arf

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Mathematics Subject Classification (2000): 11S37

DOI: 10.1007/s00229-010-0336-6

the key rôle in the construction of the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ of K. In Sects. 3 and 4, we introduce and study certain basic APF-Galois extensions $\Gamma_d^{(n)}/K$ for $1 \leq n, d \in \mathbb{Z}$, which arise naturally and are of primary importance in the theory, as

$$G_K = \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K).$$

In fact, such basic extensions $\Gamma_d^{(n)}/K$, for $1 \leq n, d \in \mathbb{Z}$, appear in the work of Laubie in [14] as well. In Sect. 5, we sketch the main steps of the construction of $\boldsymbol{\Phi}_K^{(\varphi)}$ in terms of the generalized Fesenko reciprocity maps $\boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}$ for every $1 \leq n, d \in \mathbb{Z}$. In Sect. 6, we construct the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ of K and in Sect. 7 study the basic properties of this map. Finally in Sect. 8, we sketch the construction of the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi_o)}$ of K, where K is *any* local field not necessarily satisfying $\boldsymbol{\mu}_p(K^{sep}) \subset K$, by "glueing" the non-abelian local reciprocity map $\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}$ of K_o and the local Artin map $\mathrm{Art}_{K_o/K}$ of K_o/K, where $\zeta_p \in K^{sep}$ being a fixed primitive pth root of unity, $K_o = K(\zeta_p)$ and φ_o a fixed Lubin–Tate splitting over K_o (cf. [13] for the definition of Lubin–Tate splittings).

We feel that, the present paper together with [11] and [12] should be viewed as a complement to the papers [4–6] of Fesenko. Moreover, the reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ of K introduced in this work is *closely* related to the Laubie reciprocity map $\iota_K^{(\varphi),\mathrm{Laubie}}$ of K introduced in [14]. The paper investigating the relationship of $\boldsymbol{\Phi}_K^{(\varphi)}$ with $\iota_K^{(\varphi),\mathrm{Laubie}}$ will appear elsewhere.

Finally, let us mention that, the two papers of Arf [1,2], which dates back to 1955 and 1966 respectively, studies the problem of constructing non-abelian class field theory of a local field K of positive characteristic. Although his results are preliminary at best, the method and constructions are promising. The basic idea of Arf, unfortunately never published except for a short notice [2], is to represent the action of the absolute Galois group G_K in terms of the "symbols" introduced in [1], so that we may find a concrete way of constructing a universal G_K-module $\mathcal{M}$ in the sense that the Artin representation of any finite Galois extension L/K is afforded by a canonically defined factor module of $\mathcal{M}$. Therefore, the G_K-module $\mathcal{M}$ may serve as the object in the formulation of the non-abelian local class field theory over K. It would be very interesting to relate Arf's works with the one presented in this paper.

Notation. All through this work, K will denote a local field; that is a complete discrete valuation field with finite residue class field $O_K/\mathfrak{p}_K =: \kappa_K$ of $q_K = q = p^f$ elements with p a prime number, where O_K denotes the ring of integers in K with the unique maximal ideal $\mathfrak{p}_K$. As usual, the unit group of K is denoted by U_K and the ith higher unit group of K by U_K^i, where $0 \leq i \in \mathbb{Z}$. We further assume that (except the last section) the multiplicative group $\boldsymbol{\mu}_p(K^{sep})$ of pth roots of unity in K^{sep} satisfies

$$\boldsymbol{\mu}_p(K^{sep}) \subset K. \tag{1.1}$$

Let $\boldsymbol{\nu}_K$ denote the corresponding normalized valuation on K (normalized by $\boldsymbol{\nu}_K(K^\times) = \mathbb{Z}$), and $\widetilde{\boldsymbol{\nu}}$ the unique extension of $\boldsymbol{\nu}_K$ to a fixed separable closure K^{sep} of K. For any sub-extension L/K of K^{sep}/K, the normalized form of the valuation $\widetilde{\boldsymbol{\nu}}\ |_L$ on L will be denoted by $\boldsymbol{\nu}_L$. Finally, let G_K denote the absolute Galois group $\mathrm{Gal}(K^{sep}/K)$ of K.

2. A brief review of the generalized Fesenko reciprocity map

In this section, we shall review the theory developed in [12]. Fix a Lubin–Tate splitting φ over K (cf. [13] for the definition of Lubin–Tate splittings). Let L/K be an infinite APF-Galois extension with residue degree $[\kappa_L : \kappa_K] = d$ satisfying $K \subset L \subset K_{\varphi^d}$ (i.e, L/K is a φ^d-compatible extension in the sense of [13]). As usual, the field of norms corresponding to L/K is denoted by $\mathbb{X}(L/K)$ and the completion of the maximal unramified extension $\mathbb{X}(L/K)^{nr}$ of $\mathbb{X}(L/K)$ by $\widetilde{\mathbb{X}}(L/K)$. Setting $L_0 := L_0^{(K)} = L \cap K^{nr}$, we recall that, for the extension L/K as above, the diamond subgroup $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ of the group $U_{\widetilde{\mathbb{X}}(L/K)}$ of units in the ring of integers of the local field $\widetilde{\mathbb{X}}(L/K)$ is defined by

$$U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} = \mathrm{Pr}_{\widetilde{K}}^{-1}(U_{L_0}),$$

where $\mathrm{Pr}_{\widetilde{K}} : U_{\widetilde{\mathbb{X}}(L/K)} \to U_{\widetilde{K}}$ denotes the projection map on the $\widetilde{K} = \widetilde{L}_0$-coordinate of $U_{\widetilde{\mathbb{X}}(L/K)}$ (for details, [7–9,17]).

Remark 2.1. Note that, the projection $\mathrm{Pr}_{\widetilde{K}} : U_{\widetilde{\mathbb{X}}(L/K)} \to U_{\widetilde{K}}$ is a continuous mapping. Moreover, U_{L_0} is compact. That is, U_{L_0} is closed and bounded with respect to the canonical topology induced from the valuation $\nu_{\widetilde{K}}$ of $\widetilde{K}$. Thus, $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ is closed and furthermore bounded, as

$$U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \subseteq \left\{\alpha \in \widetilde{\mathbb{X}}(L/K)^\times \mid \nu_{\widetilde{\mathbb{X}}(L/K)}(\alpha) = \nu_{\widetilde{L}_0}(\mathrm{Pr}_{\widetilde{L}_0}(\alpha)) \geq 0\right\}.$$

Now, one of the main results of [12] is the existence of a bijective 1-cocycle

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{2.1}$$

called the *generalized Fesenko reciprocity map for the extension L/K*, defined by the composition

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\boldsymbol{\phi}^{(\varphi)}_{L/K}} & K^\times/N_{L_0/K}L_0{}^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \\ & \searrow^{\boldsymbol{\Phi}^{(\varphi)}_{L/K}} & \Big\downarrow \left(\mathrm{id}_{K^\times/N_{L_0/K}L_0{}^\times}, c_{L/L_0}\right) \\ & & K^\times/N_{L_0/K}L_0{}^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \end{array} \tag{2.2}$$

where

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^{\times}/N_{L_0/K}{L_0}^{\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \tag{2.3}$$

is an injective 1-cocycle called, following [12], the *generalized arrow defined for the extension L/K*, and defined by

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma) = \left(\pi_K^m N_{L_0/K} L_0^{\times}, \phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right), \tag{2.4}$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $0 \leq m \in \mathbb{Z}$ is the integer satisfying $\sigma \mid_{L_0} = \varphi^m \mid_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$, $\pi_K = \pi_{L_o}$ is the unique prime element of K which is a universal norm in K_{φ^d}/K, and for any $\tau \in \mathrm{Gal}(L/L_0)$, the value $\phi^{(\varphi^d)}_{L/L_0}(\tau)$ of the arrow defined for the extension L/L_0 at τ is defined by [4–6] and by [11]. Namely, $\phi^{(\varphi^d)}_{L/L_0}(\tau) = U_\tau . U_{\mathbb{X}(L/L_0)}$ provided that $U_\tau \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, which is unique modulo $U_{\mathbb{X}(L/L_0)}$, solves the equation $U^{1-\varphi^d} = \Pi^{\tau-1}_{\varphi^d;L/L_0}$, where $\Pi_{\varphi^d;L/L_0}$ is the canonical prime element of the local field $\mathbb{X}(L/L_0)$ defined in Lemma 1.2 and Lemma 1.3 of [12]. For the definition of the subgroup Y_{L/L_0} of $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ satisfying the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$, whose definition is involved, we refer the reader to [6] and for more details, to the papers [4–6, 11]. In the commutative triangle (2.2), the arrow

$$c_{L/L_0} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \tag{2.5}$$

is the canonical map defined by the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$. Recall that (cf. [4–6] and [11]), the composition $c_{L/L_0} \circ \phi^{(\varphi^d)}_{L/L_0} = \Phi^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is the Fesenko reciprocity map for the extension L/L_0. Thus, for $\sigma \in \mathrm{Gal}(L/K)$, the value $\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma)$ of the generalized Fesenko reciprocity map for the extension L/K is defined by

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma) = \left(\pi_K^m N_{L_0/K} L_0^{\times}, \Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right), \tag{2.6}$$

where $0 \leq m \in \mathbb{Z}$ is the integer satisfying $\sigma \mid_{L_0} = \varphi^m \mid_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$.

Remark 2.2. The action of $\mathrm{Gal}(L/L_0)$ on $\widetilde{\mathbb{X}}(L/L_0)^{\times}$ is continuous. Therefore, it follows that $\boldsymbol{\phi}^{(\varphi^d)}_{L/L_0}$ is a continuous mapping. Moreover, as the canonical map c_{L/L_0} is continuous, the map $\boldsymbol{\Phi}^{(\varphi^d)}_{L/L_0}$ is continuous as well.

Remark 2.3. In view of Remark 2.1, both of the quotient groups $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ and $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ are compact.

Define a law of composition $*$ on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}) \cdot (\overline{b}, \overline{V})^{(\boldsymbol{\phi}^{(\varphi)}_{L/K})^{-1}((\overline{a},\overline{U}))} \tag{2.7}$$

for every $\overline{a} = a \cdot N_{L_0/K} L_0^{\times}, \overline{b} = b \cdot N_{L_0/K} L_0^{\times} \in K^{\times}/N_{L_0/K} L_0^{\times}$ with $a, b \in K^{\times}$ and $\overline{U} = U \cdot U_{\mathbb{X}(L/K)}, \overline{V} = V \cdot U_{\mathbb{X}(L/K)} \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ with $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ is defined by $(\overline{b}, \overline{V})^{\sigma} = \left(\overline{b}, \overline{V}^{\varphi^{-m}\sigma}\right)$. Then $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ is a topological group under $*$, and the map $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right), \tag{2.8}$$

where the topological group structure on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ is defined with respect to the binary operation $*$ defined by Eq. 2.7. Likewise, define a law of composition, again denoted by $*$, on $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}) \cdot (\overline{b}, \overline{V})^{(\boldsymbol{\Phi}^{(\varphi)}_{L/K})^{-1}((\overline{a}, \overline{U}))} \tag{2.9}$$

for every $\overline{a} = a \cdot N_{L_0/K} L_0^{\times}, \overline{b} = b \cdot N_{L_0/K} L_0^{\times} \in K^{\times}/N_{L_0/K} L_0^{\times}$ with $a, b \in K^{\times}$ and $\overline{U} = U \cdot Y_{L/L_0}, \overline{V} = V \cdot Y_{L/L_0} \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ with $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined by $(\overline{b}, \overline{V})^{\sigma} = \left(\overline{b}, \overline{V}^{\varphi^{-m}\sigma}\right)$. Then $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is a topological group under $*$, and the map $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{2.10}$$

where the topological group structure on $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined with respect to the binary operation $*$ defined by Eq. 2.9.

Remark 2.4. Note that $\mathrm{Gal}(L/K)$ is compact. Also the topologies on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ and on $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ are Hausdorff. Therefore, the continuity of the mappings $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ implies that $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ are closed mappings, and hence homeomorphisms.

To simplify the notation, we introduce :

Notation 2.5. Let K and L be as above. As a notation, we introduce

$$\nabla^{(\varphi)}_{L/K,U} := K^{\times}/N_{L_0/K} L_0{}^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$$

and

$$\nabla^{(\varphi)}_{L/K,Y} := K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}.$$

Furthermore, to unify the discussion, we introduce the following notation.

Notation 2.6. Let K and L be as above. Let $\Omega_{L/K}$ denote either one of the subgroups $U_{\mathbb{X}(L/K)}$ or Y_{L/L_0} of $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$. As a notation, let

$$\nabla^{(\varphi)}_{L/K,\Omega} := K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K}.$$

Remark 2.7. Let L and K be as above. In view of Remark 2.1 it follows that $\nabla^{(\varphi)}_{L/K,\Omega}$ is compact.

Then, we can define a continuous 1-cocycle,

$$\mathrm{Rec}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to \nabla^{(\varphi)}_{L/K,\Omega} \tag{2.11}$$

by

$$\mathrm{Rec}^{(\varphi)}_{L/K} = \begin{cases} \boldsymbol{\phi}^{(\varphi)}_{L/K}, & \text{if } \Omega = U; \\ \boldsymbol{\Phi}^{(\varphi)}_{L/K}, & \text{if } \Omega = Y. \end{cases} \tag{2.12}$$

Define a law of composition $*$ on $\mathrm{im}\left(\mathrm{Rec}^{(\varphi)}_{L/K}\right)$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}) \cdot (\overline{b}, \overline{V})^{(\mathrm{Rec}^{(\varphi)}_{L/K})^{-1}((\overline{a},\overline{U}))} \tag{2.13}$$

for every $\overline{a} = a \cdot N_{L_0/K}L_0^\times, \overline{b} = b \cdot N_{L_0/K}L_0^\times \in K^\times/N_{L_0/K}L_0^\times$ with $a, b \in K^\times$ and $\overline{U} = U \cdot \Omega_{L/K}, \overline{V} = V \cdot \Omega_{L/K} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K}$ with $U, V \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $\mathrm{im}\left(\mathrm{Rec}^{(\varphi)}_{L/K}\right)$ is defined by $(\overline{b}, \overline{V})^\sigma = \left(\overline{b}, \overline{V}^{\varphi^{-m}\sigma}\right)$. Then the 1-cocycle (2.11) becomes a topological group homomorphism, where the law of composition $*$ on $\mathrm{im}\left(\mathrm{Rec}^{(\varphi)}_{L/K}\right)$ is defined by (2.13).

The mapping $\mathrm{Rec}^{(\varphi)}_{L/K}$, which will simply be called the *non-abelian reciprocity map* of the extension L/K, satisfies the following basic properties.

(i) For an infinite Galois sub-extension M/K of L/K such that the residue class degree $[\kappa_M : \kappa_K] = d'$ and $K \subset M \subset K_{\varphi^{d'}}$ for some $d' \mid d$, the square

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\mathrm{Rec}^{(\varphi)}_{L/K}} & \nabla^{(\varphi)}_{L/K,\Omega} \\ \downarrow{\scriptstyle \mathrm{res}_M} & & \downarrow{\scriptstyle \left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)} \\ \mathrm{Gal}(M/K) & \xrightarrow{\mathrm{Rec}^{(\varphi)}_{M/K}} & \nabla^{(\varphi)}_{M/K,\Omega}, \end{array}$$

where the right-vertical arrow is the continuous map

$$\nabla^{(\varphi)}_{L/K,\Omega} \xrightarrow{\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)} \nabla^{(\varphi)}_{M/K,\Omega} \tag{2.14}$$

defined by

$$\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right) : (\overline{a}, \overline{U}) \mapsto \left(e^{\mathrm{CFT}}_{L_0/M_0}(\overline{a}), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})\right), \tag{2.15}$$

for every $(\overline{a}, \overline{U}) \in \nabla^{(\varphi)}_{L/K,\Omega}$, is commutative. Here, the arrow $\widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/\Omega_{M/K}$ is the Coleman norm map from L to M defined by Eqs. 2.22 and 2.23 of [12] in case $(\Omega_{L/K}, \Omega_{M/K}) = (U_{\mathbb{X}(L/K)}, U_{\mathbb{X}(M/K)})$, and defined by Lemma 2.21 together with (2.47),(2.48) of [12] in case $(\Omega_{L/K}, \Omega_{M/K}) = (Y_{L/L_0}, Y_{M/M_0})$ (also see Sect. 6), and the arrow $e^{\text{CFT}}_{L_0/M_0} : K^\times/N_{L_0/K}L_0^\times \to K^\times/N_{M_0/K}M_0^\times$ appearing in the commutative diagram is the natural inclusion defined via the existence theorem of local class field theory.

Notation 2.8. Let the local fields L and M be as above. To simplify the notation, let $\mathcal{C}_{L/M}$ denote the map $\left(e^{\text{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}\right)$ defined by (2.14) and (2.15), provided that $(\Omega_{L/K}, \Omega_{M/K}) = (U_{\mathbb{X}(L/K)}, U_{\mathbb{X}(M/K)})$ or $(Y_{L/L_0}, Y_{M/M_0})$.

(ii) Let F/K be a finite sub-extension of L/K. Put $L_0^{(F)} = L \cap F^{nr}$. Fix a Lubin–Tate splitting φ_F over F. Assume that the residue class degree $[\kappa_L : \kappa_F] = d'$ and $F \subset L \subset F_{(\varphi_F)^{d'}}$ for some $d' \mid d$. Then the square

$$\begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\mathrm{Rec}^{(\varphi_F)}_{L/F}} & \nabla^{(\varphi_F)}_{L/F,\Omega} \\ \Big\downarrow{\scriptstyle \text{inc.}} & & \Big\downarrow{\scriptstyle (N_{F/K},\lambda_{F/K})} \\ \mathrm{Gal}(L/K) & \xrightarrow{\mathrm{Rec}^{(\varphi_K)}_{L/K}} & \nabla^{(\varphi_K)}_{L/K,\Omega}, \end{array}$$

where the right-vertical arrow

$$\nabla^{(\varphi_F)}_{L/F,\Omega} \xrightarrow{(N_{F/K},\lambda_{F/K})} \nabla^{(\varphi_K)}_{L/K,\Omega} \tag{2.16}$$

defined by

$$(N_{F/K}, \lambda_{F/K}) : (\overline{a}, \overline{U}) \mapsto \left(\overline{N_{F/K}(a)}, \lambda_{F/K}(\overline{U})\right), \tag{2.17}$$

for every element $(\overline{a}, \overline{U}) \in \nabla^{(\varphi_F)}_{L/F,\Omega}$, is commutative. Here, the map $\lambda_{F/K} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)}/\Omega_{L/F} \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K}$ is defined by Eqs. 2.25–2.28 of [12] in case $(\Omega_{L/F}, \Omega_{L/K}) = (U_{\mathbb{X}(L/F)}, U_{\mathbb{X}(L/K)})$, and defined by Lemma 2.24 together with Eqs. 2.52 and 2.53 of [12] in case $(\Omega_{L/F}, \Omega_{L/K}) = (Y_{L/L_0^{(F)}}, Y_{L/L_0^{(K)}})$.

(iii) For each $0 \le i \in \mathbb{R}$, introduce the subgroups $\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^i$ of the field $\widetilde{\mathbb{X}}(L/K)$ by $\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^i = U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \cap U^i_{\widetilde{\mathbb{X}}(L/K)}$, where $U^i_{\widetilde{\mathbb{X}}(L/K)}$ is the ith higher unit group of $\widetilde{\mathbb{X}}(L/K)$. For each $0 \le n \in \mathbb{Z}$, as in Eq. 5.42 of [11], let

$$Q^n_{L/L_0} = c_{L/L_0}\left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \cap \mathrm{im}(\phi^{(\varphi^d)}_{L/L_0})\right), \tag{2.18}$$

which is a subgroup of $\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/L_0}/Y_{L/L_0}$. Here, the canonical homomorphism c_{L/L_0} introduced in (2.5) is defined by Eq. 5.35 of [11]. Now, ramification theorem for the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ corresponding to the extension L/K states, for $0 \leq n \in \mathbb{Z}$, the inclusion

$$\begin{aligned}
&\boldsymbol{\phi}^{(\varphi)}_{L/K}\left(\mathrm{Gal}(L/K)_n - \mathrm{Gal}(L/K)_{n+1}\right) \\
&\subseteq \left\langle 1_{K^\times/N_{L_0/K}L_0^\times}\right\rangle \times \left(\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}\right. \\
&\left. - \left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^{n+1} U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}\right),
\end{aligned} \tag{2.19}$$

and the ramification theorem for the generalized Fesenko reciprocity map $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ corresponding to the extension L/K states, for $0 \leq n \in \mathbb{Z}$, the inclusion

$$\begin{aligned}
&\boldsymbol{\Phi}^{(\varphi)}_{L/K}\left(\mathrm{Gal}(L/K)_n - \mathrm{Gal}(L/K)_{n+1}\right) \\
&\subseteq \left\langle 1_{K^\times/N_{L_0/K}L_0^\times}\right\rangle \times \left(\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/L_0}/Y_{L/L_0} - Q^{n+1}_{L/L_0}\right).
\end{aligned} \tag{2.20}$$

where, for $0 \leq u \in \mathbb{R}$, $\mathrm{Gal}(L/K)_u$ denotes the uth higher ramification subgroup in the lower numbering of the Galois group $\mathrm{Gal}(L/K)$ corresponding to the infinite APF-Galois extension L/K.

Finally, the following remark is in order.

Remark 2.9. We do not need assumption (1.1) on the local field K to define the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ corresponding to the extension L/K, which is introduced in (2.3) and defined by (2.4). For details, cf. [12].

3. The extensions $\Gamma_d^{(n)}$ of the local field K

In this section, we closely follow [14]. Fix a Lubin–Tate splitting φ over K. For each $1 \leq d \in \mathbb{Z}$, let K_{φ^d} denote the fixed-field of $\varphi^d \in G_K$. Observe that, $K^{sep} = K^{nr}K_{\varphi^d}$ and $K_d^{nr} = K^{nr} \cap K_{\varphi^d}$, where K_d^{nr} denotes the unique unramified extension over K of degree d.

Now, for each $1 \leq n, d \in \mathbb{Z}$, let $\Gamma_d^{(n)} := \Gamma_d^{(n)}(K, \varphi)$ be a Galois extension over K, which is the unique maximal n-abelian extension [1] of K_d^{nr} in K_{φ^d}. Note that,

Lemma 3.1. *Let $(K^{nr})^{n-ab}$ denote the "n-abelian closure" of K^{nr} in K^{sep}. Then,*

$$\bigcup_{1\leq d\in\mathbb{Z}} \Gamma_d^{(n)} = (K^{nr})^{n-ab}. \tag{3.1}$$

Proof. Look at the proof of Lemma 5 of [14]. □

Clearly, Lemma 3.1 also proves that

[1] Recall that, by an n-abelian extension over a field F, we mean a Galois extension E/F whose Galois group $\mathrm{Gal}(E/F)$ has a trivial nth-commutator subgroup $\mathrm{Gal}(E/F)^{(n)}$.

Corollary 3.2. $\bigcup_{1\le n\in\mathbb{Z}}\bigcup_{1\le d\in\mathbb{Z}}\Gamma_d^{(n)}=K^{sep}$.

Thus, we have just observed that the collection of extensions $\Gamma_d^{(n)}$ over K for $1\le n,d\in\mathbb{Z}$ are the "building blocks" of the separable closure K^{sep} of K. One of the main results of this section, stated in [14] without a proof, is the following theorem, which describes the structure of such extensions. For the sake of completeness, we shall supply a proof of this theorem.

Theorem 3.3. *For each pair (n,d) of positive integers, $\Gamma_d^{(n)}$ is an APF-extension over K.*

Proof. By class field theory, if K is any local field with finite residue field, then every abelian extension of K whose residue extension is finite is APF. Thus the assertion follows by induction from Proposition 3.4.1 of [17]. □

An immediate consequence of Theorem 3.3 is the following corollary.

Corollary 3.4. *If an extension L/K is elementwise fixed by φ^d with a finite $d\ge 1$ and L/K_d^{nr} is n-abelian, then L/K is an APF-extension.*

Proof. Clearly $L\subseteq\Gamma_d^{(n)}$. The assertion follows now from part (iii) of Lemma 3.3 of [11]. □

4. The absolute Galois group G_K of the local field K

Fix a Lubin–Tate splitting $\varphi_K=\varphi$ over K. By Corollary 3.2, the absolute Galois group G_K of the local field K is the projective limit

$$G_K=\varprojlim_{(n,d)}\mathrm{Gal}(\Gamma_d^{(n)}/K)$$

where the projective limit is defined over the restriction morphisms

$$r_{(n',d')}^{(n,d)}:\mathrm{Gal}(\Gamma_d^{(n)}/K)\to\mathrm{Gal}(\Gamma_{d'}^{(n')}/K)$$

for $(n,d),(n',d')\in\mathbb{Z}_{\ge1}\times\mathbb{Z}_{\ge1}$ satisfying $n'\le n$ and $d'\mid d$ (which is equivalent to $\Gamma_{d'}^{(n')}\subseteq\Gamma_d^{(n)}$).

5. The strategy

Now, for each $1\le n,d\in\mathbb{Z}$, $\Gamma_d^{(n)}$ is the APF-Galois extension over K which is the maximal n-abelian extension of K_d^{nr} in K_{φ^d}/K. Furthermore, the residue class degree of the extension $\Gamma_d^{(n)}/K$ is $[\kappa_{\Gamma_d^{(n)}}:\kappa_K]=d$. Therefore, the generalized Fesenko theory developed in [12] can be applied to the extensions of the form $\Gamma_d^{(n)}/K$, which would enable us to construct the *generalized arrow $\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}$ for the* *extension* $\varGamma_d^{(n)}/K$, for every pair $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ and the *generalized Fesenko reciprocity map* $\boldsymbol{\Phi}^{(\varphi)}_{\varGamma_d^{(n)}/K}$ *for the extension* $\varGamma_d^{(n)}/K$, for every pair $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

Next, as for the second step, using property (i) of the generalized arrows for the collection $\left\{\boldsymbol{\phi}^{(\varphi)}_{\varGamma_d^{(n)}/K}\right\}_{(n,d)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}}$ and using property (i) of the generalized Fesenko reciprocity maps for the collection $\left\{\boldsymbol{\Phi}^{(\varphi)}_{\varGamma_d^{(n)}/K}\right\}_{(n,d)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}}$ and passing to the projective limits, we shall construct the *generalized arrow* $\boldsymbol{\phi}_K^{(\varphi)}$ *for the local field* K and the *non-abelian local reciprocity map* $\boldsymbol{\Phi}_K^{(\varphi)}$ *for the local field* K respectively.

6. Non-abelian local reciprocity map

In this section, we shall construct the non-abelian arrow $\boldsymbol{\phi}_K^{(\varphi)}$ and the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ for the local field K.

For each pair $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, $\varGamma_d^{(n)}/K$ is an APF-Galois sub-extension of K_{φ^d}/K. Furthermore, the residue class degree of the extension $\varGamma_d^{(n)}/K$ is $[\kappa_{\varGamma_d^{(n)}} : \kappa_K] = d$. Therefore, following [12], the non-abelian reciprocity map

$$\mathrm{Rec}^{(\varphi)}_{\varGamma_d^{(n)}/K} : \mathrm{Gal}(\varGamma_d^{(n)}/K) \to \nabla^{(\varphi)}_{\varGamma_d^{(n)}/K,\varOmega} \tag{6.1}$$

for the extension $\varGamma_d^{(n)}/K$ is defined. Note that, $\varGamma_d^{(n)} \cap K^{nr} = K_d^{nr}$.

Now, suppose that $\varGamma_{d'}^{(n')} \subseteq \varGamma_d^{(n)}$. That is, equivalently, the pairs (n', d') and (n, d) satisfy $n' \leq n$ and $d' \mid d$. Then, by the 1^{st} basic property of $\mathrm{Rec}^{(\varphi)}_{\varGamma_d^{(n)}/K}$ discussed in Sect. 2, the following square

$$\begin{array}{ccc} \mathrm{Gal}(\varGamma_d^{(n)}/K) & \xrightarrow{\mathrm{Rec}^{(\varphi)}_{\varGamma_d^{(n)}/K}} & \nabla^{(\varphi)}_{\varGamma_d^{(n)}/K,\varOmega} \\ {\scriptstyle \mathrm{res}_{\varGamma_{d'}^{(n')}}}\downarrow & & \downarrow{\scriptstyle \mathcal{C}_{\varGamma_d^{(n)}/\varGamma_{d'}^{(n')}}} \\ \mathrm{Gal}(\varGamma_{d'}^{(n')}/K) & \xrightarrow{\mathrm{Rec}^{(\varphi)}_{\varGamma_{d'}^{(n')}/K}} & \nabla^{(\varphi)}_{\varGamma_{d'}^{(n')}/K,\varOmega}, \end{array} \tag{6.2}$$

is commutative.

Let us briefly recall the definition of the Coleman norm map from L to M, where L is an infinite APF-Galois extension over K satisfying $K \subset L \subset K_{\varphi^d}$ with residue class degree $[\kappa_L : \kappa_K] = d$, and M/K is an infinite Galois sub-extension of L/K satisfying $K \subset M \subset K_{\varphi^{d'}}$ with residue class degree $[\kappa_M : \kappa_K] = d'$, where $d' \mid d$. Put $L_0 = L \cap K^{nr}$ and $M_0 = M \cap K^{nr}$. Now, the Coleman norm map

$$\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\varOmega_{L/K} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/\varOmega_{M/K} \tag{6.3}$$

from L to M is defined by

$$\widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}\left(\overline{U}\right) = \widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U).\Omega_{M/K}, \tag{6.4}$$

for every $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where $\overline{U}$ denotes, as usual, the coset $U.\Omega_{L/K}$ in $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K}$. In the expression (6.4), the map $\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times}$ is defined as follows. Let

$$K \subset L_o = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

be an ascending chain satisfying $L = \bigcup_{0\leq i\in\mathbb{Z}} E_i$ and $[E_{i+1} : E_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Then

$$K \subset M_o = E_o \cap M \subseteq E_1 \cap M \subseteq \cdots \subseteq E_i \cap M \subseteq \cdots \subset M$$

is an ascending chain of field extensions satisfying the conditions $M = \bigcup_{0\leq i\in\mathbb{Z}}(E_i \cap M)$ and also $[E_{i+1}\cap M : E_i\cap M] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Thus, we construct $\mathbb{X}(M/K)$ by the sequence $(E_i\cap M)_{0\leq i\in\mathbb{Z}}$ and $\widetilde{\mathbb{X}}(M/K) = \mathbb{X}(M\widetilde{K}/\widetilde{K})$ by the sequence $(\widetilde{E_i \cap M} = M\widetilde{E}_i)_{0\leq i\in\mathbb{Z}}$. Note that, $E_i \cap M \neq E_i$ for every $0 \leq i \in \mathbb{Z}$. Furthermore, the commutative square

$$\begin{array}{ccc} \widetilde{E}_i^{\times} & \xleftarrow{\widetilde{N}_{E_{i'}/E_i}} & \widetilde{E}_{i'}^{\times} \\ {\scriptstyle \prod_{0\leq\ell\lneq f(L/M)}(\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}}\downarrow & & \downarrow{\scriptstyle \prod_{0\leq\ell\lneq f(L/M)}(\varphi^{d'})^{\ell}\widetilde{N}_{E_{i'}/E_{i'}\cap M}} \\ \widetilde{E_i \cap M}^{\times} & \xleftarrow{\widetilde{N}_{E_{i'}\cap M/E_i\cap M}} & \widetilde{E_{i'} \cap M}^{\times} \end{array}$$

for every pair $0 \leq i, i' \in \mathbb{Z}$ satisfying $i \leq i'$, induces the group homomorphism

$$\widetilde{\mathcal{N}}_{L/M} = \varprojlim_{0\leq i\in\mathbb{Z}} \left(\prod_{0\leq\ell\lneq f(L/M)} (\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M} \right) : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times} \tag{6.5}$$

which is defined by

$$\widetilde{\mathcal{N}}_{L/M}\left((\alpha_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}}\right) = \left(\prod_{0\leq\ell\lneq f(L/M)} (\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}(\alpha_{\widetilde{E}_i}) \right)_{0\leq i\in\mathbb{Z}}, \tag{6.6}$$

for every $(\alpha_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}} \in \widetilde{\mathbb{X}}(L/K)^{\times}$. For basic properties of this map, we refer the reader to [12]. The map

$$\langle\varphi\rangle_{L/M} : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(L/K)^{\times} \tag{6.7}$$

in the expression (6.4) on the other hand is defined by

$$\langle\varphi\rangle_{L/M} : \left(\alpha_{\widetilde{E}_i}\right)_{0\leq i\in\mathbb{Z}} \mapsto \left(\alpha_{\widetilde{E}_i}^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}}\right)_{0\leq i\in\mathbb{Z}}, \tag{6.8}$$

for every $\left(\alpha_{\widetilde{E}_i}\right)_{0\leq i\in\mathbb{Z}} \in \widetilde{\mathbb{X}}(L/K)^{\times}$. For basic properties of this map, we refer the reader to [12].

Remark 6.1. Let the local fields L and M be as above. For any $\alpha \in M\widetilde{K}$ and for any $\xi \in \mathrm{Gal}(L/L_0)$, where we recall that $\mathrm{Gal}(L/L_0) \simeq \mathrm{Gal}(L^{nr}/K^{nr}) \simeq \mathrm{Gal}(L\widetilde{K}/\widetilde{K})$,

$$\alpha^{\xi\varphi^{d'}} = \alpha^{\varphi^{d'}\xi},$$

as ξ acts on $M\widetilde{K}$ via its action on the M-component of $M\widetilde{K}$ and as $\varphi^{d'}$ fixes M. Thus, for $\left(\alpha_{\widetilde{E}_i}\right)_{0\leq i\in\mathbb{Z}} \in \widetilde{\mathbb{X}}(L/K)^\times$ and for $\xi \in \mathrm{Gal}(L/L_0)$, we have, for $0 \leq i \in \mathbb{Z}$,

$$\begin{aligned}\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left((\alpha_{\widetilde{E}_i})^{\xi}_{0\leq i\in\mathbb{Z}}\right)_i &= \widetilde{N}_{E_i/E_i\cap M}(\alpha_{\widetilde{E}_i})^{\xi(1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)})^2} \\ &= \widetilde{N}_{E_i/E_i\cap M}(\alpha_{\widetilde{E}_i})^{(1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)})^2\xi} \\ &= \widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left((\alpha_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}}\right)_i^{\xi|_M},\end{aligned}$$

which proves that

$$\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left((\alpha_{\widetilde{E}_i})^{\xi}_{0\leq i\in\mathbb{Z}}\right) = \widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left((\alpha_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}}\right)^{\xi|_M}. \quad (6.9)$$

This property was not mentioned in [11,12].

Lemma 6.2. *Let the local fields L and M be as above. Let F/K be an infinite Galois sub-extension of M/K satisfying $K \subset F \subset K_{\varphi^{d''}}$ with residue class degree $[\kappa_F : \kappa_K] = d''$ where $d'' \mid d'$. Put $F_0 = F \cap K^{nr}$. Then the following equalities hold.*

(i) $\mathcal{C}_{L/M} = id$*, if* $L = M$.
(ii) $\mathcal{C}_{L/F} = \mathcal{C}_{M/F} \circ \mathcal{C}_{L/M}$.

Proof. Part (i) is trivial. Now, for part (ii), let $(\overline{a}, \overline{U}) \in K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K}$, where $a \in K^\times$ and $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ and as usual, $\overline{a}$ denotes the coset $a \cdot N_{L_0/K}L_0^\times$ in $K^\times/N_{L_0/K}L_0^\times$ and $\overline{U}$ denotes the coset $U.\Omega_{L/K}$ in $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/\Omega_{L/K}$. Then,

$$\mathcal{C}_{M/F} \circ \mathcal{C}_{L/M}(\overline{a}, \overline{U}) = \left(e^{\mathrm{CFT}}_{M_0/F_0} \circ e^{\mathrm{CFT}}_{L_0/M_0}(\overline{a}), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{M/F} \circ \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})\right),$$

where $e^{\mathrm{CFT}}_{M_0/F_0} \circ e^{\mathrm{CFT}}_{L_0/M_0} = e^{\mathrm{CFT}}_{L_0/F_0}$ by the existence theorem of local class field theory. For the second coordinate $\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{M/F} \circ \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})$, direct computation yields

$$\begin{aligned}\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{M/F} \circ \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U}) &= \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{M/F}\left(\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U)\Omega_{M/K}\right) \\ &= \widetilde{\mathcal{N}}_{M/F} \circ \langle\varphi\rangle_{M/F}\left(\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U)\right)\Omega_{F/K} \\ &= \widetilde{\mathcal{N}}_{M/F} \circ \widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{M/F} \circ \langle\varphi\rangle_{L/M}(U)\Omega_{F/K} \\ &= \widetilde{\mathcal{N}}_{L/F} \circ \langle\varphi\rangle_{L/F}(U)\Omega_{F/K} \\ &= \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/F}(\overline{U}),\end{aligned}$$

as $\widetilde{\mathcal{N}}_{M/F} \circ \widetilde{\mathcal{N}}_{L/M} = \widetilde{\mathcal{N}}_{L/F}$, $\langle\varphi\rangle_{M/F} \circ \langle\varphi\rangle_{L/M} = \langle\varphi\rangle_{L/F}$, and by the general equality

$$\widetilde{\mathcal{N}}_{L/M}(U^{1+\varphi^{d''}+\cdots+\varphi^{d''(f(M/F)-1)}}) = \widetilde{\mathcal{N}}_{L/M}(U)^{1+\varphi^{d''}+\cdots+\varphi^{d''(f(M/F)-1)}}$$

for $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$. Thus, the desired equality $\mathcal{C}_{L/F} = \mathcal{C}_{M/F} \circ \mathcal{C}_{L/M}$ follows now. □

Therefore, it follows that,

Corollary 6.3. *The system*

$$\left\{\nabla^{(\varphi)}_{\Gamma_d^{(n)}/K,\Omega}; \mathcal{C}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}\right\}_{\substack{n'\le n\\ d'|d}} \tag{6.10}$$

is projective.

Let

$$\begin{aligned}
\nabla^{(\varphi)}_{K,\Omega} = \nabla_{K,\Omega} &= \varprojlim_{(n,d)} \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K,\Omega} \\
&= \varprojlim_{(n,d)} K^{\times}/N_{K_d^{nr}/K} K_d^{nr\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/\Omega_{\Gamma_d^{(n)}/K} \\
&= \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/\Omega_{\Gamma_d^{(n)}/K} \\
&= \begin{cases} \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)}, & \text{if } \Omega = U \\ \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/Y_{\Gamma_d^{(n)}/K_d^{nr}}, & \text{if } \Omega = Y \end{cases}
\end{aligned} \tag{6.11}$$

be the projective limit of the system (6.10). The limit $\nabla^{(\varphi)}_{K,\Omega}$ or $\nabla_{K,\Omega}$, if there is no risk of confusion, depends on the choice of a Lubin–Tate splitting φ over K.

Note that, by Remark 2.7, it follows that the projective limit $\nabla^{(\varphi)}_{L/K,\Omega}$ is compact. Moreover by Remarks 2.2 and 6.1, $\nabla_{K,\Omega}$ has a natural topological G_K-module structure, where the G_K-action on $\nabla_{K,\Omega}$ is defined by

$$\left((\overline{a}_{d,n}, \overline{U}_{d,n})\right)^{\sigma}_{d,n} = \left((\overline{a}_{d,n}, \overline{U}_{d,n})^{\sigma|_{\Gamma_d^{(n)}}}\right)_{d,n} \tag{6.12}$$

for every coherent sequence $\left((\overline{a}_{d,n}, \overline{U}_{d,n})\right)_{d,n}$ from $\nabla_{K,\Omega}$ and, following Sect. 4, for every $\sigma \in G_K = \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$.

Now, following Sect. 4, the commutative square (6.2) together with Remark 2.2 then induces a continuous injective map

$$\mathrm{Rec}^{(\varphi)}_K = \varprojlim_{(n,d)} \mathrm{Rec}^{(\varphi)}_{\Gamma_d^{(n)}/K} : G_K \to \nabla_{K,\Omega}, \tag{6.13}$$

defined by

$$\mathrm{Rec}^{(\varphi)}_K\left((\sigma_{d,n})_{d,n}\right) = \left(\mathrm{Rec}^{(\varphi)}_{\Gamma_d^{(n)}/K}(\sigma_{d,n})\right)_{d,n}, \tag{6.14}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K) = G_K$.

Remark 6.4. Therefore, there exists a continuous injective map

$$\boldsymbol{\phi}_K^{(\varphi)} = \varprojlim_{(n,d)} \boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)} : G_K \to \nabla_{K,U}, \tag{6.15}$$

defined by

$$\boldsymbol{\phi}_K^{(\varphi)}\left((\sigma_{d,n})_{d,n}\right) = \left(\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}(\sigma_{d,n})\right)_{d,n}, \tag{6.16}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K) = G_K$, and a continuous bijective map

$$\boldsymbol{\Phi}_K^{(\varphi)} = \varprojlim_{(n,d)} \boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)} : G_K \to \nabla_{K,Y}, \tag{6.17}$$

defined by

$$\boldsymbol{\Phi}_K^{(\varphi)}\left((\sigma_{d,n})_{d,n}\right) = \left(\boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}(\sigma_{d,n})\right)_{d,n}, \tag{6.18}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K) = G_K$.

Moreover, the main feature of the mapping $\mathrm{Rec}_K^{(\varphi)}$ is the following.

Proposition 6.5. *The injective mapping* $\mathrm{Rec}_K^{(\varphi)} : G_K \to \nabla_{K,\Omega}$ *is a continuous* 1*-cocycle. That is, for* $\sigma, \tau \in G_K$ *with corresponding coherent sequences* $(\sigma_{d,n})_{d,n}, (\tau_{d,n})_{d,n} \in \varprojlim_{(n,d)} Gal(\Gamma_d^{(n)}/K)$ *respectively,*

$$\mathrm{Rec}_K^{(\varphi)}(\sigma\tau) = \mathrm{Rec}_K^{(\varphi)}(\sigma)\mathrm{Rec}_K^{(\varphi)}(\tau)^{\sigma}. \tag{6.19}$$

Proof. Follows from the topological G_K-module structure of $\nabla_{K,\Omega}$ defined by (6.12). Straightforward and trivial. □

Therefore, the 1-cocycle $\mathrm{Rec}_K^{(\varphi)} : G_K \to \nabla_{K,\Omega}$ induces a topological group isomorphism

$$\mathrm{Rec}_K^{(\varphi)} : G_K \xrightarrow{\sim} \mathrm{im}\left(\mathrm{Rec}_K^{(\varphi)}\right), \tag{6.20}$$

where the group structure on $\mathrm{im}\left(\mathrm{Rec}_K^{(\varphi)}\right)$ is defined by the binary operation

$$* : \mathrm{im}\left(\mathrm{Rec}_K^{(\varphi)}\right) \times \mathrm{im}\left(\mathrm{Rec}_K^{(\varphi)}\right) \to \mathrm{im}\left(\mathrm{Rec}_K^{(\varphi)}\right) \tag{6.21}$$

given by

$$\mathrm{Rec}_K^{(\varphi)}(\sigma) * \mathrm{Rec}_K^{(\varphi)}(\tau) = \mathrm{Rec}_K^{(\varphi)}(\sigma)\mathrm{Rec}_K^{(\varphi)}(\tau)^{\sigma}, \tag{6.22}$$

for every $\sigma, \tau \in G_K$.

Remark 6.6. Following the same lines of reasoning of Remark 2.4, it follows that $\mathrm{Rec}_K^{(\varphi)} : G_K \to \nabla_{K,\Omega}$ is an injective, closed and continuous mapping. Therefore, $\mathrm{Rec}_K^{(\varphi)} : G_K \xrightarrow{\sim} \mathrm{im}\left(\mathrm{Rec}_K^{(\varphi)}\right)$ is a topological group isomorphism.

Furthermore, we have the following lemma.

Lemma 6.7. *Let the local fields L and M be as above. The map*

$$\mathcal{C}_{L/M} : im\left(\mathrm{Rec}_{L/K}^{(\varphi)}\right) \to im\left(\mathrm{Rec}_{M/K}^{(\varphi)}\right)$$

is a group homomorphism, where the group operations on im $\left(\mathrm{Rec}_{L/K}^{(\varphi)}\right)$ *and on im* $\left(\mathrm{Rec}_{M/K}^{(\varphi)}\right)$ *are defined by Eq. 2.13.*

Proof. Note that, commutativity of the square in (i) of Sect. 2 defines the arrow $\mathcal{C}_{L/M} : \mathrm{im}\left(\mathrm{Rec}_{L/K}^{(\varphi)}\right) \to \mathrm{im}\left(\mathrm{Rec}_{M/K}^{(\varphi)}\right)$ in a well-defined manner. Moreover, for $\sigma_1, \sigma_2 \in \mathrm{Gal}(L/K)$,

$$\begin{aligned} &\mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_1) * \mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_2)\right) \\ &= \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_1) \cdot \mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_2)^{\sigma_1}\right) \\ &= \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_1)\right) \cdot \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_2)^{\sigma_1}\right) \\ &= \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_1)\right) \cdot \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_2)\right)^{\sigma_1|_M} \end{aligned}$$

which follows from Remark 6.1 and by the fact that $\Omega_{L/K}$ (and respectively $\Omega_{M/K}$) is a $\mathrm{Gal}(L/L_0)$-module (and respectively a $\mathrm{Gal}(M/M_0)$-module). Thus,

$$\mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_1) * \mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_2)\right) = \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_1)\right) * \mathcal{C}_{L/M}\left(\mathrm{Rec}_{L/K}^{(\varphi)}(\sigma_2)\right)$$

completing the proof. □

Thus, by Lemma 6.2 and by Lemma 6.7,

Corollary 6.8. *The system*

$$\left\{im\left(\mathrm{Rec}_{\Gamma_d^{(n)}/K}^{(\varphi)}\right); \mathcal{C}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}\right\}_{\substack{n' \leq n \\ d' | d}} \tag{6.23}$$

is projective.

Now, the projective limit $\varprojlim_{(n,d)} \mathrm{im}(\mathrm{Rec}_{\Gamma_d^{(n)}/K}^{(\varphi)})$ of the system (6.23) is a topological group under the law of composition $*$ on $\varprojlim_{(n,d)} \mathrm{im}(\mathrm{Rec}_{\Gamma_d^{(n)}/K}^{(\varphi)})$ defined by

$$\begin{aligned} &\left(\mathrm{Rec}_{\Gamma_d^{(n)}}^{(\varphi)}(\sigma_{d,n})\right)_{d,n} * \left(\mathrm{Rec}_{\Gamma_d^{(n)}}^{(\varphi)}(\tau_{d,n})\right)_{d,n} \\ &= \left(\mathrm{Rec}_{\Gamma_d^{(n)}}^{(\varphi)}(\sigma_{d,n}) * \mathrm{Rec}_{\Gamma_d^{(n)}}^{(\varphi)}(\tau_{d,n})\right)_{d,n} \end{aligned} \tag{6.24}$$

for every coherent sequences $(\sigma_{d,n})_{d,n}, (\tau_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$. It is now straightforward to prove the following lemma.

Lemma 6.9. *There exists a natural isomorphism*

$$im\left(\mathrm{Rec}_K^{(\varphi)}\right) \simeq \varprojlim_{(n,d)} im\left(\mathrm{Rec}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right) \tag{6.25}$$

of topological groups, where the group law $*$ *on* $im\left(\mathrm{Rec}_K^{(\varphi)}\right)$ *is defined by* (6.22) *and the group law* $*$ *on* $\varprojlim_{(n,d)} im\left(\mathrm{Rec}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)$ *is defined by* (6.24).

Remark 6.10. Note that, for (n,d) and (n',d') satisfying $n' \leq n$ and $d' \mid d$, the following square

$$\begin{array}{ccc} U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)} & \xrightarrow{c_{\Gamma_d^{(n)}/K_d^{nr}}} & U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/Y_{\Gamma_d^{(n)}/K_d^{nr}} \\ \Big\downarrow \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}} & & \Big\downarrow \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}} \\ U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_{d'}^{(n')}/K)}/U_{\mathbb{X}(\Gamma_{d'}^{(n')}/K)} & \xrightarrow{c_{\Gamma_{d'}^{(n')}/K_{d'}^{nr}}} & U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_{d'}^{(n')}/K)}/Y_{\Gamma_{d'}^{(n')}/K_{d'}^{nr}} \end{array} \tag{6.26}$$

is commutative, where $c_{L/L_0} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ denotes the canonical mapping defined by the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$ and introduced in (2.5). Therefore, by Lemma 6.2 and the commutativity of the diagram (6.26) yield a topological G_K-module homomorphism

$$c_K := \varprojlim_{(n,d)} (\mathrm{id}_{K^\times/N_{K_d^{nr}/K}K_d^{nr\times}}, c_{\Gamma_d^{(n)}/K_d^{nr}}) : \nabla_{K,U} \to \nabla_{K,Y} \tag{6.27}$$

which, by commutative triangle (2.2) and by the Property (i) of the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and of the generalized Fesenko map $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$, connects the arrows $\boldsymbol{\phi}^{(\varphi)}_K$ and $\boldsymbol{\Phi}^{(\varphi)}_K$ as

$$\boldsymbol{\Phi}^{(\varphi)}_K : G_K \xrightarrow{\boldsymbol{\phi}^{(\varphi)}_K} \nabla_{K,U} \xrightarrow{c_K} \nabla_{K,Y}. \tag{6.28}$$

The main theorem of this paper, which has already been proven in this section, can be summarized as follows.

Theorem 6.11. *There exists an injective mapping*

$$\boldsymbol{\phi}^{(\varphi)}_K : G_K \to \nabla^{(\varphi)}_{K,U}$$

satisfying the 1*-cocycle condition*

$$\boldsymbol{\phi}^{(\varphi)}_K(\sigma\tau) = \boldsymbol{\phi}^{(\varphi)}_K(\sigma)\boldsymbol{\phi}^{(\varphi)}_K(\tau)^{\sigma},$$

for every $\sigma, \tau \in G_K$*. This mapping* $\boldsymbol{\phi}_K^{(\varphi)} : G_K \to \nabla_{K,U}^{(\varphi)}$ *induces a topological group isomorphism*

$$\boldsymbol{\phi}_K^{(\varphi)} : G_K \xrightarrow{\sim} im\left(\boldsymbol{\phi}_K^{(\varphi)}\right),$$

and called the generalized arrow of K*, where the law of composition* $*$ *on* $im\left(\boldsymbol{\phi}_K^{(\varphi)}\right)$ *is defined by*

$$\boldsymbol{\phi}_K^{(\varphi)}(\sigma) * \boldsymbol{\phi}_K^{(\varphi)}(\tau) = \boldsymbol{\phi}_K^{(\varphi)}(\sigma)\boldsymbol{\phi}_K^{(\varphi)}(\tau)^{\sigma},$$

for every $\sigma, \tau \in G_K$*. Moreover, the composition*

$$\boldsymbol{\Phi}_K^{(\varphi)} : G_K \xrightarrow{\boldsymbol{\phi}_K^{(\varphi)}} \nabla_{K,U}^{(\varphi)} \xrightarrow{c_K} \nabla_{K,Y}^{(\varphi)},$$

where the topological G_K*-module homomorphism* $c_K : \nabla_{K,U}^{(\varphi)} \to \nabla_{K,Y}^{(\varphi)}$ *is defined by*

$$c_K := \varprojlim_{(n,d)} (id_{K^\times/N_{K_d^{nr}/K}K_d^{nr\times}}, c_{\Gamma_d^{(n)}/K_d^{nr}}) : \nabla_{K,U}^{(\varphi)} \to \nabla_{K,Y}^{(\varphi)},$$

induces a bijective mapping

$$\boldsymbol{\Phi}_K^{(\varphi)} : G_K \to \nabla_{K,Y}^{(\varphi)}$$

satisfying the 1*-cocycle condition*

$$\boldsymbol{\Phi}_K^{(\varphi)}(\sigma\tau) = \boldsymbol{\Phi}_K^{(\varphi)}(\sigma)\boldsymbol{\Phi}_K^{(\varphi)}(\tau)^{\sigma},$$

for every $\sigma, \tau \in G_K$*. The map* $\boldsymbol{\Phi}_K^{(\varphi)} : G_K \to \nabla_{K,Y}^{(\varphi)}$ *induces an isomorphism of topological groups*

$$\boldsymbol{\Phi}_K^{(\varphi)} : G_K \xrightarrow{\sim} \nabla_{K,Y}^{(\varphi)},$$

which is called the non-abelian local reciprocity map of K*, where the topological group structure on* $\nabla_{K,Y}^{(\varphi)}$ *is defined with respect to the binary operation* $*$ *defined by*

$$\begin{aligned}\left((\overline{a}_{d,n}, \overline{U}_{d,n})\right)_{d,n} * \left((\overline{b}_{d,n}, \overline{V}_{d,n})\right)_{d,n} &= \left((\overline{a}_{d,n}, \overline{U}_{d,n}) * (\overline{b}_{d,n}, \overline{V}_{d,n})\right)_{d,n} \\ &= \left((\overline{a}_{d,n}, \overline{U}_{d,n})\right)_{d,n} \left((\overline{b}_{d,n}, \overline{V}_{d,n})\right)_{d,n}^{(\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\left(((\overline{a}_{d,n}, \overline{U}_{d,n}))_{d,n}\right)},\end{aligned}$$

for every coherent sequences $\left((\overline{a}_{d,n}, \overline{U}_{d,n})\right)_{d,n}, \left((\overline{b}_{d,n}, \overline{V}_{d,n})\right)_{d,n} \in \nabla_{K,Y}^{(\varphi)}$.

Finally, the following remark is in order.

Remark 6.12. By Remark 2.9, we do not need assumption (1.1) on the local field K to define the generalized arrow $\boldsymbol{\phi}_K^{(\varphi)}$ of K, which is introduced in (6.15) and defined by (6.16).

7. Basic properties of the non-abelian local reciprocity map

The non-abelian local reciprocity map

$$\boldsymbol{\Phi}_K^{(\varphi)} : G_K \xrightarrow{\sim} \nabla_{K,Y}$$

of K has the following basic properties.

Proposition 7.1.

$$\boldsymbol{\Phi}_K^{(\varphi)}(W_K) = \mathbb{Z} \times \varprojlim_{(n,d)} U^{\diamond}_{\tilde{\mathbb{X}}(\Gamma_d^{(n)}/K)} / Y_{\Gamma_d^{(n)}/K_d^{nr}}, \tag{7.1}$$

where W_K denotes the Weil group of K.

Proof. Straightforward. In fact, by the definition of the Weil group of K, the following diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\Phi}_K^{(\varphi)}(I_K) & \longrightarrow & \nabla_{K,Y} & \xrightarrow{\mathrm{pr}_1} & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \boldsymbol{\Phi}_K^{(\varphi)}} & & \uparrow{\scriptstyle \boldsymbol{\Phi}_K^{(\varphi)}} & & \| & & \\
1 & \longrightarrow & I_K & \longrightarrow & G_K & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \| & & \uparrow{\scriptstyle \mathrm{inc.}} & & \uparrow{\scriptstyle \mathrm{inc.}} & & \\
1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

is commutative. Therefore, it follows that $\mathrm{pr}_1^{-1}(\mathbb{Z}) = \boldsymbol{\Phi}_K^{(\varphi)}(W_K)$, which completes the proof. □

Let F be a finite φ-compatible extension of the local field K. We fix the Lubin–Tate splitting $\varphi_F = \varphi^{f(F/K)}$ over F. Then clearly $K_\varphi \subseteq F_{\varphi_F}$. Now, choose any infinite APF-Galois extension L' over K with residue class degree $[\kappa_{L'} : \kappa_K] = d'$ and which satisfies $K \subset L' \subset K_{\varphi^{d'}}$ and the inclusion $K \subset F \subset L'$. Let L/K be an infinite Galois subextension of L'/K such that the residue class degree $[\kappa_L : \kappa_K] = d$ and which satisfies $K \subset L \subset K_{\varphi^d}$ and the inclusion $K \subset F \subset L$. Then, clearly L' is an infinite APF-Galois extension over F with residue class degree $[\kappa_{L'} : \kappa_F] = \frac{d'}{f(F/K)}$ and satisfying $F \subset L' \subset F_{\varphi_F^{\frac{d'}{f(F/K)}}} = K_{\varphi^{d'}}$, and L is an infinite APF-Galois extension over F with residue class degree $[\kappa_L : \kappa_F] = \frac{d}{f(F/K)}$ and satisfying $F \subset L \subset F_{\varphi_F^{\frac{d}{f(F/K)}}} = K_{\varphi^d}$.

Remark 7.2. Alternatively, we could have chosen the chain of fields extensions, $K \subseteq F \subset L \subset L'$ in such a way that, F is a finite φ-compatible extension of the local field K, L'/K a Galois extension such that L'/F is APF-Galois with $[\kappa_{L'} : \kappa_F] = d_o$ satisfying $L' \subset F_{\varphi_F^{d_o}}$, and L/K is a Galois extension such that L/F is APF-Galois with $[\kappa_L : \kappa_F] = d_1$ satisfying $L \subset F_{\varphi_F^{d_1}}$, where as usual $\varphi_F = \varphi^{f(F/K)}$.

Under such a choice of valued fields $K \subset F \subset L \subset L'$ (or choosing the chain $K \subset F \subset L \subset L'$ as in Remark 7.2), we have the following lemma.

Lemma 7.3. *The following square*

$$\begin{array}{ccc} \nabla^{(\varphi_F)}_{L'/F,Y} & \xrightarrow{\mathcal{C}_{L'/L}} & \nabla^{(\varphi_F)}_{L/F,Y} \\ {\scriptstyle (N_{F/K},\lambda_{F/K})}\downarrow & & \downarrow{\scriptstyle (N_{F/K},\lambda_{F/K})} \\ \nabla^{(\varphi_K)}_{L'/K,Y} & \xrightarrow{\mathcal{C}_{L'/L}} & \nabla^{(\varphi_K)}_{L/K,Y} \end{array} \tag{7.2}$$

is commutative.

Proof. Let $(\overline{a}, \overline{U}) \in \nabla^{(\varphi_F)}_{L'/F,Y}$, where $\overline{a} = a \cdot N_{L'^{(F)}_o/F} L'^{(F)\times}_o$ with $a \in F^\times$ and $\overline{U} = U \cdot Y_{L'/L'^{(F)}_o}$ with $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L'/F)}$. Then

$$\begin{aligned} \mathcal{C}_{L'/L}((\overline{a}, \overline{U})) &= \left(e^{CFT}_{L'^{(F)}_o/L^{(F)}_o}(\overline{a}), \widetilde{\mathcal{N}}^{Coleman}_{L'/L}(\overline{U})\right) \\ &= \left(a \cdot N_{L^{(F)}_o/F} L^{(F)\times}_o, \widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}(U).Y_{L/L^{(F)}_o}\right). \end{aligned}$$

and

$$\begin{aligned} &(N_{F/K}, \lambda_{F/K})\left(\left(a \cdot N_{L^{(F)}_o/F} L^{(F)\times}_o, \widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}(U).Y_{L/L^{(F)}_o}\right)\right) \\ &\quad = \left(N_{F/K}(a) \cdot N_{L^{(K)}_o/K} L^{(K)\times}_o, \lambda_{F/K}(\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}(U) \cdot Y_{L/L^{(F)}_o})\right). \end{aligned}$$

On the other hand,

$$(N_{F/K}, \lambda_{F/K})((\overline{a}, \overline{U})) = \left(N_{F/K}(a) \cdot N_{L'^{(K)}_o/K} L'^{(K)\times}_o, \lambda_{F/K}(\overline{U})\right)$$

and

$$\begin{aligned} &\mathcal{C}_{L'/L}\left(\left(N_{F/K}(a) \cdot N_{L'^{(K)}_o/K} L'^{(K)\times}_o, \lambda_{F/K}(\overline{U})\right)\right) \\ &\quad = \left(e^{CFT}_{L'^{(K)}_o/L^{(K)}_o}, \widetilde{\mathcal{N}}^{Coleman}_{L'/L}\right)\left(\left(N_{F/K}(a) \cdot N_{L'^{(K)}_o/K} L'^{(K)\times}_o, \lambda_{F/K}(\overline{U})\right)\right) \\ &\quad = \left(N_{F/K}(a) \cdot N_{L^{(K)}_o/K} L^{(K)\times}_o, \widetilde{\mathcal{N}}^{Coleman}_{L'/L}(\lambda_{F/K}(\overline{U}))\right). \end{aligned}$$

Thus, to complete the proof, it suffices to show that for $\overline{U} = U \cdot Y_{L'/L'^{(F)}_o}$ with $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L'/F)}$,

$$\lambda_{F/K}(\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}(U) \cdot Y_{L/L^{(F)}_o}) = \widetilde{\mathcal{N}}^{Coleman}_{L'/L}(\lambda_{F/K}(\overline{U})).$$

Now, following Lemma 2.24 together with Eqs. 2.52 and 2.53 of [12],

$$\lambda_{F/K}(\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}(U) \cdot Y_{L/L^{(F)}_o}) = \Lambda_{F/K}\left(\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}(U)\right) Y_{L/L^{(K)}_o}$$

and

$$\widetilde{\mathcal{N}}^{Coleman}_{L'/L}(\lambda_{F/K}(\overline{U})) = \widetilde{\mathcal{N}}^{Coleman}_{L'/L}(\Lambda_{F/K}(U)Y_{L'/L'^{(K)}_o})$$
$$= \widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi\rangle_{L'/L}\,(\Lambda_{F/K}(U))Y_{L/L^{(K)}_o}.$$

Let

$$F \subset L'^{(F)}_o = E'_o \subset E'_1 \subset \cdots \subset E'_i \subset \cdots \subset L'$$

be an ascending chain satisfying $L' = \bigcup_{0\leq i\in\mathbb{Z}} E'_i$ and $[E'_{i+1} : E'_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Now, setting $U = (u_{\widetilde{E'}_i})_{0\leq i\in\mathbb{Z}}$,

$$\Lambda_{F/K} : U = (u_{\widetilde{E'}_i})_{0\leq i\in\mathbb{Z}} \mapsto (\widetilde{N}_{L'^{(F)}_o/L'^{(K)}_o}(u_{\widetilde{E'}_0}); u_{\widetilde{E'}_i})_{0\leq i\in\mathbb{Z}}.$$

As $[\kappa_L : \kappa_F] = \frac{d}{f(F/K)}$, it follows that $\langle\varphi_F\rangle_{L'/L}\,(U) = \left(u^{\Sigma}_{\widetilde{E'}_i}\right)_{0\leq i\in\mathbb{Z}}$, where $\Sigma = 1 + \varphi_F^{\frac{d}{f(F/K)}} + \cdots + \varphi_F^{\frac{d}{f(F/K)}(f(L'/L)-1)} = 1 + \varphi^d + \cdots + \varphi^{d(f(L'/L)-1)}$. Therefore,

$$\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}\,(U) = \widetilde{\mathcal{N}}_{L'/L}\left(\left(u^{\Sigma}_{\widetilde{E'}_i}\right)_{0\leq i\in\mathbb{Z}}\right)$$
$$= \left(\prod_{0\leq \ell \lneq f(L'/L)} (\varphi_F^{\frac{d}{f(F/K)}})^{\ell}\,\widetilde{N}_{E'_i/E'_i\cap L}(u^{\Sigma}_{\widetilde{E'}_i})\right)_{0\leq i\in\mathbb{Z}}$$
$$= \left(\prod_{0\leq \ell \lneq f(L'/L)} (\varphi^{d})^{\ell}\,\widetilde{N}_{E'_i/E'_i\cap L}(u^{\Sigma}_{\widetilde{E'}_i})\right)_{0\leq i\in\mathbb{Z}}$$

and

$$\Lambda_{F/K}\left(\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi_F\rangle_{L'/L}\,(U)\right) = \Lambda_{F/K}\left(\prod_{0\leq \ell \lneq f(L'/L)} (\varphi^{d})^{\ell}\,\widetilde{N}_{E'_i/E'_i\cap L}(u^{\Sigma}_{\widetilde{E'}_i})\right)_{0\leq i\in\mathbb{Z}}$$
$$= \left(\widetilde{N}_{L^{(F)}_o/L^{(K)}_o}(\prod_{0\leq \ell \lneq f(L'/L)} (\varphi^{d})^{\ell}\,\widetilde{N}_{L'^{(F)}_o/L^{(F)}_o}(u^{\Sigma}_{\widetilde{E'}_0})); \prod_{0\leq \ell \lneq f(L'/L)} (\varphi^{d})^{\ell}\,\widetilde{N}_{E'_i/E'_i\cap L}(u^{\Sigma}_{\widetilde{E'}_i})\right)_{0\leq i\in\mathbb{Z}}.$$

On the other hand,

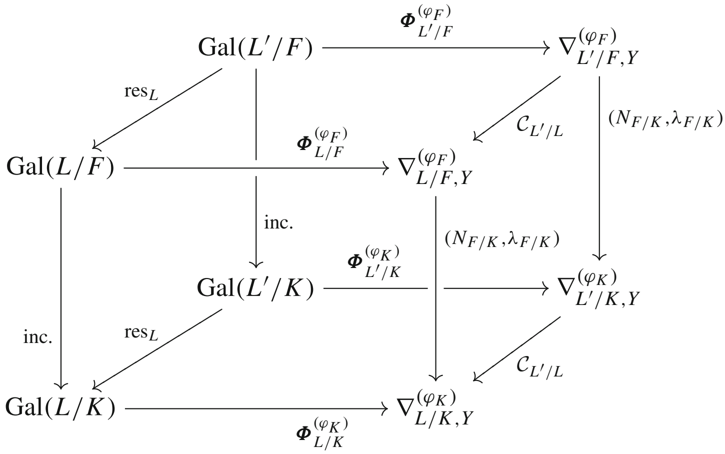
$$\widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi\rangle_{L'/L}\,(\Lambda_{F/K}(U)) = \widetilde{\mathcal{N}}_{L'/L} \circ \langle\varphi\rangle_{L'/L}\left((\widetilde{N}_{L'^{(F)}_o/L'^{(K)}_o}(u_{\widetilde{E'}_0}); u_{\widetilde{E'}_i})_{0\leq i\in\mathbb{Z}}\right)$$
$$= \widetilde{\mathcal{N}}_{L'/L}\left((\widetilde{N}_{L'^{(F)}_o/L'^{(K)}_o}(u_{\widetilde{E'}_0})^{\Sigma}; u^{\Sigma}_{\widetilde{E'}_i})_{0\leq i\in\mathbb{Z}}\right)$$
$$= \left(\prod_{0\leq \ell \lneq f(L'/L)} (\varphi^{d})^{\ell}\,\widetilde{N}_{L'^{(K)}_o/L^{(K)}_o}\left(\widetilde{N}_{L'^{(F)}_o/L'^{(K)}_o}(u_{\widetilde{E'}_0})^{\Sigma}\right); \prod_{0\leq \ell \lneq f(L'/L)} (\varphi^{d})^{\ell}\,\widetilde{N}_{E'_i/E'_i\cap L}(u^{\Sigma}_{\widetilde{E'}_i})\right)_{0\leq i\in\mathbb{Z}}.$$

Now, comparing the $\widetilde{L_o^{(K)}}$-coordinates, we have the equality

$$\widetilde{N}_{L_o^{(F)}/L_o^{(K)}}\left(\prod_{0\leq \ell \lneq f(L'/L)} (\varphi^d)^\ell \widetilde{N}_{L_o'^{(F)}/L_o^{(F)}}\left(u^{\Sigma}_{\widetilde{E'}_0}\right)\right) = \prod_{0\leq \ell \lneq f(L'/L)} (\varphi^d)^\ell \widetilde{N}_{L_o'^{(K)}/L_o^{(K)}}\left(\widetilde{N}_{L_o'^{(F)}/L_o'^{(K)}}(u_{\widetilde{E'}_0})^{\Sigma}\right),$$

and this completes the proof. □

Thus, by Lemma 7.3, for the chain of extensions of valued fields $K \subseteq F \subset L \subset L'$, the following diagram



(7.3)

is commutative. Thus, passing to the projective limits over $\{L/K\}$ and $\{L/F\}$, the commutative diagram (7.3) induces the following commutative square

$$\begin{CD} G_F @>{\boldsymbol{\Phi}_F^{(\varphi_F)}}>> \nabla_{F,Y}^{(\varphi_F)} \\ @V{\text{inc.}}VV @VV{\mathcal{N}_{F/K}^{\infty}}V \\ G_K @>{\boldsymbol{\Phi}_K^{(\varphi)}}>> \nabla_{K,Y}^{(\varphi)}, \end{CD} \tag{7.4}$$

where we recall that F/K is a *finite* extension *compatible* with respect to the Lubin–Tate splitting $\varphi = \varphi_K$ over K. From the commutativity of the square (7.4), transitivity

$$\mathcal{N}_{F'/K}^{\infty} = \mathcal{N}_{F/K}^{\infty} \circ \mathcal{N}_{F'/F}^{\infty} \tag{7.5}$$

follows, where $K \subseteq F \subseteq F'$ is a tower of field extensions of finite degree such that F/K and F'/K (and hence F'/F) are compatible with φ. As a notation, let

$$\mathcal{N}_{F/K}^{\infty}(\nabla_{F,Y}^{(\varphi_F)}) =: \mathcal{N}_F^{\infty}, \tag{7.6}$$

which is a *closed* subgroup of $\nabla_{K,Y}^{(\varphi)}$. If L/K is an infinite separable extension, which is a union of certain finite extensions E over K compatible with φ, we introduce the closed subgroup $\mathcal{N}_L^{\infty}$ of $\nabla_{K,Y}^{(\varphi)}$ by

$$\mathcal{N}_L^{\infty} = \bigcap_E \mathcal{N}_E^{\infty}, \tag{7.7}$$

where E/K runs over *all* finite φ-compatible subextensions of L/K.

Lemma 7.4. *Let L/K be a Galois extension which is a union of certain finite extensions E over K compatible with φ. Then the surjective and continuous group homomorphism*

$$\nabla_{K,Y}^{(\varphi)} \quad \underbrace{\xrightarrow[\sim]{(\boldsymbol{\Phi}_K^{(\varphi)})^{-1}}}_{\text{non-ab. "norm-residue" hom.}} \quad G_K \xrightarrow{res_L} Gal(L/K) \tag{7.8}$$

has kernel

$$\ker\left(res_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right) = \mathcal{N}_L^{\infty}, \tag{7.9}$$

and induces the continuous isomorphism

$$\nabla_{K,Y}^{(\varphi)}/\mathcal{N}_L^{\infty} \xrightarrow{\sim} Gal(L/K). \tag{7.10}$$

Proof. Let L/K be a *finite* Galois extension compatible with respect to φ. First, note that

$$\ker\left(\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right) = \left\{\delta \in \nabla_{K,Y}^{(\varphi)} : (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}(\delta) \in G_L\right\}.$$

Therefore, for any $\delta \in \ker\left(\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right)$, there exists $\xi \in \nabla_{L,Y}^{(\varphi_L)}$, such that $(\boldsymbol{\Phi}_L^{(\varphi_L)})^{-1}(\xi) = (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}(\delta)$. Thus, it follows that $\mathcal{N}_{L/K}^{\infty}(\xi) = \boldsymbol{\Phi}_K^{(\varphi)} \circ (\boldsymbol{\Phi}_L^{(\varphi_L)})^{-1}(\xi) = \delta$ by the commutativity of the square (7.4). Thus, we have the inclusion

$$\ker\left(\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right) \subseteq \mathcal{N}_{L/K}^{\infty}(\nabla_{L,Y}^{(\varphi_L)}).$$

The proof of the reverse inclusion is similar. Therefore,

$$\nabla_{K,Y}^{(\varphi)}/\mathcal{N}_L^{\infty} \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

Now, assume that L/K is *any* Galois extension, which is a union of finite extensions E over K compatible with φ. Then, clearly, for any $K \subseteq E \subseteq L$, where E/K is a finite extension compatible with φ, the inclusion

$$\ker\left(\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right) \subseteq \mathcal{N}_E^{\infty}, \tag{7.11}$$

follows, as $G_L \subseteq G_E$. Therefore,

$$\ker\left(\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right) \subseteq \bigcap_E \mathcal{N}_E^{\infty} = \mathcal{N}_L^{\infty}.$$

To prove the reverse inclusion, let $\delta \in \bigcap_E \mathcal{N}_E^{\infty} = \mathcal{N}_L^{\infty}$. Thus, for each finite subextension E/K in L/K compatible with φ, there exists $\xi_E \in \nabla_{E,Y}^{(\varphi_E)}$, such that $\mathcal{N}_{E/K}^{\infty}(\xi_E) = \delta$. Now, our aim is to show that $\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}(\delta) = \mathrm{id}_L$. In fact, this immediately follows, as for each E, $\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}(\delta) = \mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\left(\mathcal{N}_{E/K}^{\infty}(\xi_E)\right) = \mathrm{res}_L \circ (\boldsymbol{\Phi}_E^{(\varphi_E)})^{-1}(\xi_E) \in \mathrm{Gal}(L/E)$. Thus, the reverse inclusion

$$\mathcal{N}_L^{\infty} \subseteq \ker\left(\mathrm{res}_L \circ (\boldsymbol{\Phi}_K^{(\varphi)})^{-1}\right)$$

follows. Therefore,

$$\nabla_{K,Y}^{(\varphi)}/\mathcal{N}_L^{\infty} \xrightarrow{\sim} \mathrm{Gal}(L/K),$$

and the proof is now complete. □

Now, if L/K is *any* finite Galois extension, then L has a finite unramified extension L' such that the extension L'/K is compatible with φ_K (cf. 0.4 in [13]). If this is the case, there exists the following commutative diagram

$$\begin{array}{ccc}
G_{L'} & \xrightarrow{\boldsymbol{\Phi}_{L'}^{(\varphi_{L'})}} & \nabla_{L',Y}^{(\varphi_{L'})} \\
\downarrow{\scriptstyle \mathrm{inc.}} & & \vdots \\
G_L & & \vdots\ {\scriptstyle \mathcal{N}_{L'/K}^{\infty}} \\
\downarrow{\scriptstyle \mathrm{inc.}} & & \downarrow \\
G_K & \xrightarrow{\boldsymbol{\Phi}_K^{(\varphi)}} & \nabla_{K,Y}^{(\varphi)},
\end{array} \tag{7.12}$$

Therefore, define the closed subgroup $\mathcal{N}_L^{\infty}$ of $\nabla_{K,Y}^{(\varphi_K)}$ by

$$\mathcal{N}_L^{\infty} := \boldsymbol{\Phi}_K^{(\varphi_K)}(G_L), \tag{7.13}$$

and if L/K is an infinite Galois extension, define the closed subgroup $\mathcal{N}_L^{\infty}$ of $\nabla_{K,Y}^{(\varphi_K)}$ by

$$\mathcal{N}_L^{\infty} = \bigcap_E \mathcal{N}_E^{\infty}, \tag{7.14}$$

where E/K runs over all finite Galois subextensions of L/K. Then, the existence theorem of non-abelian local class field theory, whose proof is now straightforward, can be stated as follows.

Proposition 7.5 (Existence Theorem). *Let L/K be a Galois extension. Fix Lubin–Tate splittings φ_L over L and φ_K over K. Then the surjective and continuous group homomorphism*

$$\nabla_{K,Y}^{(\varphi_K)} \underbrace{\xrightarrow[\sim]{(\boldsymbol{\Phi}_K^{(\varphi_K)})^{-1}}}_{\text{non-ab. “norm-residue” hom.}} G_K \xrightarrow{res_L} Gal(L/K) \tag{7.15}$$

has kernel

$$\ker\left(res_L \circ (\boldsymbol{\Phi}_K^{(\varphi_K)})^{-1}\right) = \mathcal{N}_{L/K}^{\infty}\left(\nabla_{L,Y}^{(\varphi_L)}\right) =: \mathcal{N}_L^{\infty}, \tag{7.16}$$

and induces the continuous isomorphism

$$\nabla_{K,Y}^{(\varphi_K)}/\mathcal{N}_L^{\infty} \xrightarrow{\sim} Gal(L/K). \tag{7.17}$$

Moreover, there is a bijection between all Galois extensions over K and all closed subgroups of $\nabla_{K,Y}^{(\varphi_K)}$ defined by

$$L/K \mapsto \mathcal{N}_L^{\infty} \tag{7.18}$$

for every Galois extension L/K. The group $\mathcal{N}_L^{\infty}$ is of finite index in $\nabla_{K,Y}^{(\varphi_K)}$ if and only if L/K is finite Galois, and if this is the case $[L:K] = (\nabla_{K,Y}^{(\varphi_K)} : \mathcal{N}_L^{\infty})$.

Let $\sigma : K \hookrightarrow K^{sep}$ be any embedding of K in K^{sep} and let $\widetilde{\sigma} \in \mathrm{Aut}(K^{sep})$ be any extension to K^{sep} of the embedding $\sigma : K \hookrightarrow K^{sep}$. Following the same lines of reasoning as in [10], the automorphism $\widetilde{\sigma}$ of K^{sep} defines an isomorphism of topological groups

$$\widetilde{\sigma}^{+} : \nabla_{K,Y}^{(\varphi)} \xrightarrow{\sim} \nabla_{K^{\sigma},Y}^{(\widetilde{\sigma}\varphi\widetilde{\sigma}^{-1})} \tag{7.19}$$

and we have the following “Galois conjugation law” for the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ of K.

Proposition 7.6 (Galois Conjugation). *For any $\alpha \in \nabla_{K,Y}^{(\varphi)}$,*

$$\left(\boldsymbol{\Phi}_{K^{\sigma}}^{(\widetilde{\sigma}\varphi\widetilde{\sigma}^{-1})}\right)^{-1}(\widetilde{\sigma}^{+}(\alpha)) = \widetilde{\sigma} \circ \left(\boldsymbol{\Phi}_K^{(\varphi)}\right)^{-1}(\alpha) \circ \widetilde{\sigma}^{-1}. \tag{7.20}$$

Proof. Follows from the same lines of reasoning as in [10]. □

8. General case

The aim of this section, in which we follow closely the "profinite version" of [3], is to sketch the extension of the theory for *any* local field K (that is, K does not necessarily satisfy (1.1)). If this is the case, then setting $K_o = K(\zeta_p)$, where ζ_p is any primitive pth root of unity in K^{sep}, the local field K_o satisfies (1.1). Moreover, the following sequence

$$1 \to G_{K_o} \xrightarrow{i} G_K \xrightarrow{\mathrm{res}_{K_o}} \mathrm{Gal}(K_o/K) \to 1 \tag{8.1}$$

of topological groups is exact. This group extension induces, via the conjugation action of G_K on G_{K_o} (which yields a group homomorphism $\psi_o : G_K/G_{K_o} \to \mathrm{Out}(G_{K_o})$), a group homomorphism $\psi : \mathrm{Gal}(K_o/K) \xrightarrow{\overline{\mathrm{res}}_{K_o}^{-1}} G_K/G_{K_o} \xrightarrow{\psi_o} \mathrm{Out}(G_{K_o}) = \mathrm{Aut}(G_{K_o})/\mathrm{Inn}(G_{K_o})$, where $\mathrm{Inn}(G_{K_o})$ is the subgroup of $\mathrm{Aut}(G_{K_o})$ consisting of all inner automorphisms of G_{K_o} and $\mathrm{Out}(G_{K_o})$ is the outer automorphism group of G_{K_o}, which is the cokernel of the canonical mapping $\alpha : G_{K_o} \to \mathrm{Aut}(G_{K_o})$. This exact sequence is in the category of profinite groups. Therefore, by [16],

Theorem 8.1. *Given the topological short exact sequence* (8.1). *Then, there exists a continuous section*

$$s : Gal(K_o/K) \to G_K \tag{8.2}$$

of the morphism $G_K \to Gal(K_o/K)$, which can be assumed, without loss of generality, to be "normalized" in the sense that $s(id_{K_o}) = id_K$.

Now, define the mapping

$$f : \mathrm{Gal}(K_o/K) \times \mathrm{Gal}(K_o/K) \to G_{K_o} \tag{8.3}$$

satisfying the equation

$$s(\tau)s(\tau') = f(\tau, \tau')s(\tau\tau'), \tag{8.4}$$

for every $\tau, \tau' \in \mathrm{Gal}(K_o/K)$. Also, let us denote the lifting of the homomorphism $\psi : \mathrm{Gal}(K_o/K) \to \mathrm{Out}(G_{K_o})$ from $\mathrm{Out}(G_{K_o})$ to $\mathrm{Aut}(G_{K_o})$, determined by (8.2), by $\psi^* : \mathrm{Gal}(K_o/K) \to \mathrm{Aut}(G_{K_o})$. Then the following identities hold.

$$\psi^*(\tau)\psi^*(\tau') = \alpha(f(\tau, \tau'))\psi^*(\tau\tau'), \tag{8.5}$$

for every $\tau, \tau' \in \mathrm{Gal}(K_o/K)$, and the cocycle condition

$$f(\tau, \tau')f(\tau\tau', \tau'') = \psi^*(f(\tau', \tau''))f(\tau, \tau'\tau''), \tag{8.6}$$

for every $\tau, \tau', \tau'' \in \mathrm{Gal}(K_o/K)$. Now, define a law of composition on $E_{f,\psi^*} = G_{K_o} \times \mathrm{Gal}(K_o/K)$ by

$$(\gamma, \tau)(\gamma', \tau') = \left(\gamma\psi^*(\tau)(\gamma')f(\tau, \tau'), \tau\tau'\right), \tag{8.7}$$

for every $(\gamma, \tau), (\gamma', \tau') \in E_{f,\psi^*}$. Then, E_{f,ψ^*} under the law of composition (8.7) is a topological group, and sits in the topological short exact sequence

$$1 \to G_{K_o} \xrightarrow{\iota} E_{f,\psi^*} \xrightarrow{\mathrm{Pr}_2} \mathrm{Gal}(K_o/K) \to 1. \tag{8.8}$$

The proof of the following theorem, on the "center triviality" of the local absolute Galois groups, can be found in the work of Mochizuki [15].

Theorem 8.2. *For any local field L, the center $Z(G_L)$ of the absolute Galois group G_L of L is trivial.*

Therefore, G_{K_o} has a trivial center. So, fixing a homomorphism $\psi : \mathrm{Gal}(K_o/K) \to \mathrm{Out}(G_{K_o})$, there exists *exactly one* topological extension of $\mathrm{Gal}(K_o/K)$ by G_{K_o} up to equivalence. Thus, there exists a topological group isomorphism

$$\rho : G_K \xrightarrow{\sim} E_{f,\psi^*} \tag{8.9}$$

which sits in the following

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_{K_o} & \xrightarrow{i} & G_K & \xrightarrow{\mathrm{res}_{K_o}} & \mathrm{Gal}(K_o/K) & \longrightarrow & 1 \\
 & & \| & & \wr\downarrow\rho & & \| & & \\
1 & \longrightarrow & G_{K_o} & \xrightarrow{\iota} & E_{f,\psi^*} & \xrightarrow{\mathrm{Pr}_2} & \mathrm{Gal}(K_o/K) & \longrightarrow & 1
\end{array} \tag{8.10}$$

commutative diagram. Let $\rho_* : G_K/G_{K_o} \xrightarrow{\sim} E_{f,\psi^*}/\iota(G_{K_o})$ be the isomorphism induced by the commutative diagram (8.10). Then the homomorphism $\psi' : \mathrm{Gal}(K_o/K) \to \mathrm{Out}(G_{K_o})$ induced by the group extension $1 \to G_{K_o} \xrightarrow{\iota} E_{f,\psi^*} \xrightarrow{\mathrm{Pr}_2} \mathrm{Gal}(K_o/K) \to 1$ is defined by the composition $\psi' = \iota_* \circ \psi'_o \circ \overline{\mathrm{Pr}_2}^{-1}$: $\mathrm{Gal}(K_o/K) \xrightarrow[\overline{\mathrm{Pr}_2}^{-1}]{\sim} E_{f,\psi^*}/\iota(G_{K_o}) \xrightarrow{\psi'_o} \mathrm{Out}(\iota(G_{K_o})) \xrightarrow[\iota_*]{\sim} \mathrm{Out}(G_{K_o})$. Here, the isomorphism $\iota_* : \mathrm{Out}(\iota(G_{K_o})) \xrightarrow{\sim} \mathrm{Out}(G_{K_o})$ is defined naturally by $\iota_* : \overline{\xi} \mapsto \overline{\iota^{-1} \circ \xi \circ \iota}$, for every $\xi \in \mathrm{Aut}(\iota(G_{K_o}))$. The following diagram

$$\begin{array}{ccccc}
G_K/G_{K_o} & & \xrightarrow{\psi_o} & & \mathrm{Out}(G_{K_o}) \\
\rho_*^{-1}\uparrow\downarrow\rho_* & \nwarrow^{\overline{\mathrm{res}}_{K_o}^{-1}} & \mathrm{Gal}(K_o/K) & \nearrow^{\psi}\ \nearrow^{\psi'} & \iota_*\uparrow\downarrow\iota_*^{-1} \\
 & \swarrow_{\overline{\mathrm{Pr}_2}^{-1}} & & & \\
E_{f,\psi^*}/\iota(G_{K_o}) & & \xrightarrow[\psi'_o]{} & & \mathrm{Out}(\iota(G_{K_o}))
\end{array}$$

is commutative. Thus, there are different presentations of the homomorphism ψ' : $\mathrm{Gal}(K_o/K) \to \mathrm{Out}(G_{K_o})$. Note that the mapping

$$s' : \mathrm{Gal}(K_o/K) \to E_{f,\psi^*} \tag{8.11}$$

defined by

$$s' : \tau \mapsto \rho(s(\tau)), \tag{8.12}$$

for every $\tau \in \mathrm{Gal}(K_o/K)$ defines a continuous normalized section of the projection mapping $\mathrm{Pr}_2 : E_{f,\psi^*} \to \mathrm{Gal}(K_o/K)$. Therefore, the mapping

$$f' : \mathrm{Gal}(K_o/K) \times \mathrm{Gal}(K_o/K) \to G_{K_o} \tag{8.13}$$

defined, with respect to $s' : \mathrm{Gal}(K_o/K) \to E_{f,\psi^*}$, by

$$s'(\tau)s'(\tau') = \iota(f'(\tau, \tau'))s'(\tau\tau'), \tag{8.14}$$

for every $\tau, \tau' \in \mathrm{Gal}(K_o/K)$, satisfies

$$\iota \circ f'(\tau, \tau') = \rho \circ f(\tau, \tau'), \tag{8.15}$$

for every $\tau, \tau' \in \mathrm{Gal}(K_o/K)$. Also, the lifting

$$\psi'^* : \mathrm{Gal}(K_o/K) \to \mathrm{Aut}(G_{K_o}) \tag{8.16}$$

of the homomorphism $\psi' : \mathrm{Gal}(K_o/K) \to \mathrm{Out}(G_{K_o})$ from $\mathrm{Out}(G_{K_o})$ to $\mathrm{Aut}(G_{K_o})$ determined by the map (8.11) defined by (8.12) is essentially the mapping ψ^* and is given by

$$\psi'^*(\tau) = \iota^{-1} \circ \rho \circ \psi^*(\tau) \circ \rho^{-1} \circ \iota, \tag{8.17}$$

for every $\tau \in \mathrm{Gal}(K_o/K)$. Thus, the law of composition on $G_{K_o} \times \mathrm{Gal}(K_o/K)$ defined by

$$(\gamma, \tau)(\gamma', \tau') = (\gamma.\psi'^*(\tau)(\gamma')f'(\tau, \tau'), \tau\tau'),$$

for every $(\gamma, \tau), (\gamma', \tau') \in G_{K_o} \times \mathrm{Gal}(K_o/K)$, is nothing but the group operation on $G_{K_o} \times \mathrm{Gal}(K_o/K)$ defined by (8.7).

Now, fix a Lubin–Tate splitting φ_o over the local field K_o. As the local field K_o satisfies (1.1), the non-abelian local reciprocity map

$$\boldsymbol{\Phi}_{K_o}^{(\varphi_o)} : G_{K_o} \xrightarrow{\sim} \nabla_{K_o,Y}$$

of K_o exists. Let $\mathrm{Art}_{K_o/K} : \mathrm{Gal}(K_o/K) \xrightarrow{\sim} K^\times/N_{K_o/K}K_o^\times$ be the local Artin reciprocity map for the abelian extension K_o/K. Now, our aim is to introduce a law of composition $*$ on $\nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times$ so that, $\nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times$ becomes a topological group under the binary operation $*$, and the bijective map

$$\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right) : E_{f,\psi^*} \to \nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times \tag{8.18}$$

defined by

$$\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)((\gamma, \tau)) = \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}(\gamma), \mathrm{Art}_{K_o/K}(\tau)\right) \tag{8.19}$$

for every $(\gamma, \tau) \in E_{f,\psi^*}$ becomes a topological group isomorphism which makes the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{K_o} & \xrightarrow{\iota} & E_{f,\psi^*} & \xrightarrow{\mathrm{Pr}_2} & \mathrm{Gal}(K_o/K) & \longrightarrow & 1 \\ & & \downarrow \wr\, \boldsymbol{\Phi}_{K_o}^{(\varphi_o)} & & \downarrow \wr\, \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right) & & \downarrow \wr\, \mathrm{Art}_{K_o/K} & & \\ 1 & \longrightarrow & \nabla_{K_o,Y} & \xrightarrow{\iota} & \nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times & \xrightarrow{\mathrm{Pr}_2} & K^\times/N_{K_o/K}K_o^\times & \longrightarrow & 1 \end{array} \tag{8.20}$$

commutative. In order to do so, define the group homomorphism

$$\boldsymbol{\psi} : K^\times/N_{K_o/K}K_o^\times \to \mathrm{Out}(\nabla_{K_o,Y}) \tag{8.21}$$

by the composition

$$\boldsymbol{\psi} : K^\times/N_{K_o/K}K_o^\times \xrightarrow[\sim]{\mathrm{Art}_{K_o/K}^{-1}} \mathrm{Gal}(K_o/K) \xrightarrow{\psi'} \mathrm{Out}(G_{K_o}) \xrightarrow[\sim]{\mathsf{T}_{\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}}} \mathrm{Out}(\nabla_{K_o,Y}), \tag{8.22}$$

where

$$\mathsf{T}_{\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}} : \mathrm{Out}(G_{K_o}) \xrightarrow{\sim} \mathrm{Out}(\nabla_{K_o,Y}) \tag{8.23}$$

is the well-defined group isomorphism defined, for every $f \in \mathrm{Aut}(G_{K_o})$, by

$$\begin{aligned} &\mathsf{T}_{\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}} : f \circ \mathrm{Inn}(G_{K_o}) = \overline{f} \mapsto \\ &\quad (\boldsymbol{\Phi}_{K_o}^{(\varphi_o)} \circ f \circ (\boldsymbol{\Phi}_{K_o}^{(\varphi_o)})^{-1}) \circ \mathrm{Inn}(\nabla_{K_o,Y}) = \overline{\boldsymbol{\Phi}_{K_o}^{(\varphi_o)} \circ f \circ (\boldsymbol{\Phi}_{K_o}^{(\varphi_o)})^{-1}}. \end{aligned} \tag{8.24}$$

Now, define the continuous mapping

$$\boldsymbol{s} : K^\times/N_{K_o/K}K_o^\times \to \nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times \tag{8.25}$$

by

$$\boldsymbol{s}(\tau) = \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right) \circ s' \circ \mathrm{Art}_{K_o/K}^{-1}(\tau), \tag{8.26}$$

for every $\tau \in K^\times/N_{K_o/K}K_o^\times$. Clearly, this map is a continuous and normalized section of the morphism $\mathrm{Pr}_2 : \nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times \to K^\times/N_{K_o/K}K_o^\times$, because for $\tau \in K^\times/N_{K_o/K}K_o^\times$, setting $s'(\mathrm{Art}_{K_o/K}^{-1}(\tau)) = (\gamma, \mathrm{Art}_{K_o/K}^{-1}(\tau))$ for some $\gamma \in G_{K_o}$, it follows that $\mathrm{Pr}_2 \circ \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)((\gamma, \mathrm{Art}_{K_o/K}^{-1}(\tau))) = \tau$. Therefore, we can define the continuous map

$$\boldsymbol{f} : K^\times/N_{K_o/K}K_o^\times \times K^\times/N_{K_o/K}K_o^\times \to \nabla_{K_o,Y}, \tag{8.27}$$

with respect to $\boldsymbol{s} : K^\times/N_{K_o/K}K_o^\times \to \nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times$, by

$$\boldsymbol{s}(\tau)\boldsymbol{s}(\tau') = \iota(\boldsymbol{f}(\tau, \tau'))\boldsymbol{s}(\tau\tau'), \tag{8.28}$$

for every $\tau, \tau' \in K^\times/N_{K_o/K}K_o^\times$; and also the lifting

$$\boldsymbol{\psi}^* : K^\times/N_{K_o/K}K_o^\times \to \mathrm{Aut}(\nabla_{K_o,Y}) \tag{8.29}$$

of the homomorphism $\boldsymbol{\psi} : K^{\times}/N_{K_o/K}K_o^{\times} \to \text{Out}(\nabla_{K_o,Y})$ from $\text{Out}(\nabla_{K_o,Y})$ to $\text{Aut}(\nabla_{K_o,Y})$ determined by the map (8.25) defined by (8.26). Thus, under the law of composition $*$ on $\nabla_{K_o,Y} \times K^{\times}/N_{K_o/K}K_o^{\times}$ defined by

$$(\eta, \tau) * (\eta', \tau') = (\eta * \boldsymbol{\psi}^*(\tau)(\eta') * \boldsymbol{f}(\tau, \tau'), \tau\tau'), \tag{8.30}$$

for every $(\eta, \tau), (\eta', \tau') \in \nabla_{K_o,Y} \times K^{\times}/N_{K_o/K}K_o^{\times}$, $\nabla_{K_o,Y} \times K^{\times}/N_{K_o/K}K_o^{\times}$ is a topological group, and sits in the topological short exact sequence

$$1 \to \nabla_{K_o,Y} \xrightarrow{\iota} \nabla_{K_o,Y} \times K^{\times}/N_{K_o/K}K_o^{\times} \xrightarrow{\text{Pr}_2} \text{Gal}(K_o/K) \to 1. \tag{8.31}$$

Theorem 8.3. *The bijection*

$$\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, Art_{K_o/K}\right) : E_{f,\psi^*} \to \nabla_{K_o,Y} \times K^{\times}/N_{K_o/K}K_o^{\times}$$

defined by

$$\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, Art_{K_o/K}\right)((\gamma, \tau)) = \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}(\gamma), Art_{K_o/K}(\tau)\right),$$

for every $(\gamma, \tau) \in E_{f,\psi^*}$, is a topological group isomorphism and makes the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_{K_o} & \xrightarrow{\iota} & E_{f,\psi^*} & \xrightarrow{Pr_2} & Gal(K_o/K) & \longrightarrow & 1 \\
& & \boldsymbol{\Phi}_{K_o}^{(\varphi_o)} \downarrow \wr & & \wr \downarrow \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, Art_{K_o/K}\right) & & \wr \downarrow Art_{K_o/K} & & \\
1 & \longrightarrow & \nabla_{K_o} & \longrightarrow & \nabla_{K_o,Y} \times K^{\times}/N_{K_o/K}K_o^{\times} & \xrightarrow{Pr_2} & K^{\times}/N_{K_o/K}K_o^{\times} & \longrightarrow & 1
\end{array}$$

commutative.

Proof. For $(\gamma, \tau), (\gamma', \tau') \in E_{f,\psi^*}$, the following basic equalities

$$\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}\left(\psi'^*(\tau)(\gamma')\right) = \boldsymbol{\psi}^*\left(\text{Art}_{K_o/K}(\tau)\right)\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}(\gamma')\right)$$

and

$$\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}\left(f'(\tau, \tau')\right) = \boldsymbol{f}\left(\text{Art}_{K_o/K}(\tau), \text{Art}_{K_o/K}(\tau')\right)$$

are satisfied. In fact, by the definition of (8.29),

$$\boldsymbol{\psi}^*\left(\text{Art}_{K_o/K}(\tau)\right) = \boldsymbol{\Phi}_{K_o}^{(\varphi_o)} \circ \psi'^*(\tau) \circ (\boldsymbol{\Phi}_{K_o}^{(\varphi_o)})^{-1},$$

which yields the first basic equality

$$\boldsymbol{\psi}^*\left(\text{Art}_{K_o/K}(\tau)\right)\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}(\gamma')\right) = \boldsymbol{\Phi}_{K_o}^{(\varphi_o)} \circ \psi'^*(\tau) \circ (\boldsymbol{\Phi}_{K_o}^{(\varphi_o)})^{-1}\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}(\gamma')\right).$$

Now, for the second basic equality, let $\xi, \xi' \in K^{\times}/N_{K_o/K}{K_o}^{\times}$ such that $\text{Art}_{K_o/K}(\tau) = \xi$ and $\text{Art}_{K_o/K}(\tau') = \xi'$. Then, by (8.25) and (8.26),

$$\begin{aligned}
\boldsymbol{s}(\xi)\boldsymbol{s}(\xi') &= \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \text{Art}_{K_o/K}\right) \circ s'(\tau)\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \text{Art}_{K_o/K}\right) \circ s'(\tau') \\
&= \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \text{Art}_{K_o/K}\right)(s'(\tau)s'(\tau')),
\end{aligned}$$

and it follows from (8.13) and (8.14) that,

$$\begin{aligned}\boldsymbol{s}(\xi)\boldsymbol{s}(\xi') &= \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)(\iota(f'(\tau,\tau'))s'(\tau\tau'))\\ &= \left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)(\iota(f'(\tau,\tau')))\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)(s'(\tau\tau')).\end{aligned}$$

On the other hand, by (8.27) and (8.28),

$$\boldsymbol{s}(\xi)\boldsymbol{s}(\xi') = \iota(\boldsymbol{f}(\xi,\xi'))\boldsymbol{s}(\xi\xi') = \iota(\boldsymbol{f}(\xi,\xi'))\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)\circ s'(\tau\tau').$$

Thus, we end up having the identity

$$\left(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}\right)(\iota(f'(\tau,\tau'))) = \iota(\boldsymbol{f}(\xi,\xi')) = \iota(\boldsymbol{f}(\mathrm{Art}_{K_o/K}(\tau), \mathrm{Art}_{K_o/K}(\tau'))),$$

which yields the second basic equality

$$\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}(f'(\tau,\tau')) = \boldsymbol{f}(\mathrm{Art}_{K_o/K}(\tau), \mathrm{Art}_{K_o/K}(\tau')).$$

Now, the proof of the theorem follows from these two basic equalities. □

Thus, by Theorem 8.3, there exists a topological isomorphism

$$\boldsymbol{\Phi}_{K}^{(\varphi_o)} : G_K \xrightarrow{\sim} \nabla_{K_o,Y}^{(\varphi_o)} \times K^\times/N_{K_o/K}{K_o}^\times \tag{8.32}$$

defined by the composition

$$\boldsymbol{\Phi}_{K}^{(\varphi_o)} : G_K \underset{\sim}{\xrightarrow{\rho}} E_{f,\psi'} \underset{\sim}{\xrightarrow{(\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K})}} \nabla_{K_o,Y}^{(\varphi_o)} \times K^\times/N_{K_o/K}K_o^\times, \tag{8.33}$$

and called the *non-abelian local reciprocity map of* K. Moreover, the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_{K_o} & \xrightarrow{\;i\;} & G_K & \xrightarrow{\;\mathrm{res}_{K_o}\;} & \mathrm{Gal}(K_o/K) & \longrightarrow & 1\\
 & & \Vert & & \wr\downarrow \rho & & \Vert & & \\
1 & \longrightarrow & G_{K_o} & \xrightarrow{\;\iota\;} & E_{f,\psi^*} & \xrightarrow{\;\mathrm{Pr}_2\;} & \mathrm{Gal}(K_o/K) & \longrightarrow & 1\\
 & & \boldsymbol{\Phi}_{K_o}^{(\varphi_o)}\downarrow\wr & & \wr\downarrow (\boldsymbol{\Phi}_{K_o}^{(\varphi_o)}, \mathrm{Art}_{K_o/K}) & & \wr\downarrow \mathrm{Art}_{K_o/K} & & \\
1 & \longrightarrow & \nabla_{K_o} & \xrightarrow{\;\iota\;} & \nabla_{K_o,Y} \times K^\times/N_{K_o/K}K_o^\times & \xrightarrow{\;\mathrm{Pr}_2\;} & K^\times/N_{K_o/K}K_o^\times & \longrightarrow & 1
\end{array} \tag{8.34}$$

is commutative. Therefore, we can "glue" the non-abelian local class field theory over K_o and the abelian local class field theory for the cyclotomic extension K_o/K to get the basic properties of the non-abelian local reciprocity map $\boldsymbol{\Phi}_{K}^{(\varphi_o)} : G_K \to \nabla_{K_o,Y}^{(\varphi_o)} \times K^\times/N_{K_o/K}K_o^\times$, which is the non-abelian local class field theory over K.

Acknowledgements. The first named author (K. I. I.) is partly supported by the TÜBİTAK grant 107T728. Both of the authors would like to thank D. Burns, D. Delbourgo, and especially to I. B. Fesenko for their interest and encouragement at all stages of this work.

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