**Ramification theory in non-abelian local class field theory**

by

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*To the memory of I. M. Gelfand*

1. **Introduction.** Let $K$ be a local field, that is, a complete discrete valuation field with finite residue class field $\kappa_K$ of $q = p^f$ elements. For technical reasons, throughout the paper we shall assume that the multiplicative group $\mu_p(K^{\text{sep}})$ of all $p$th roots of unity in $K^{\text{sep}}$ satisfies $\mu_p(K^{\text{sep}}) \subset K$. Fix a Lubin–Tate splitting $\varphi$ over $K$. That is, we fix an extension $\varphi$ of the Frobenius automorphism of $K^{\text{nr}}$ to $K^{\text{sep}}$ (for details, cf. [Ko-dS]). In a sequence of papers [Ik-Se-1, Ik-Se-2, Ik-Se-3], following the idea of Fesenko developed in [Fes-1, Fes-2, Fes-3], we have constructed the non-abelian local reciprocity map $\Phi_K^{(\varphi)}$ for $K$, which is an isomorphism from the absolute Galois group $G_K$ of $K$ onto a certain topological group $\nabla_K^{(\varphi)}$ which depends on the choice of the Lubin–Tate splitting $\varphi$.

The aim of the present paper is to study the ramification-theoretic properties of the map $\Phi_K^{(\varphi)}$. We prove (in Theorems 4.15 and 4.16) that $\Phi_K^{(\varphi)}$ is compatible with the refined higher ramification “filtration” of the absolute Galois group $G_K$ of $K$ (cf. 4.1) and the refined “filtration” of $\nabla_K^{(\varphi)}$ (cf. 4.2).

The organization of the paper is as follows. In Section 2 we collect the necessary results from the theory of local fields. In Section 3 we briefly review the main results of [Ik-Se-2] on the generalized Fesenko reciprocity map, and then sketch the construction of the non-abelian local reciprocity map $\Phi_K^{(\varphi)}$ following [Ik-Se-3]. In the last section, we first introduce the refined filtrations on $G_K$ and on $\nabla_K^{(\varphi)}$ and then prove the main results of the paper, which are stated as Theorems 4.15 and 4.16.

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2. Preliminaries on local fields. In this section, we shall briefly review the necessary background material from the theory of local fields.

2.1. Local fields. Throughout this work, $K$ will denote a local field, that is, a complete discrete valuation field with finite residue class field $O_K/p_K =: \kappa_K$ of $q_K = q = p^f$ elements with $p$ a prime number, where $O_K$ denotes the ring of integers in $K$ with the unique maximal ideal $p_K$. Let $\nu_K$ denote the corresponding normalized valuation on $K$ (normalized by $\nu_K(K^\times) = \mathbb{Z}$). As usual, the unit group of $K$ is denoted by $U_K$ and the $i$th higher unit group of $K$ by $U^i_K$, where $0 \leq i \in \mathbb{Z}$.

Let $K_{\text{sep}}$ denote a fixed separable closure of $K$, and $K_{\text{nr}}$ the maximal unramified extension of $K$ inside $K_{\text{sep}}$. The unique extension of $\nu_K$ to $K_{\text{sep}}$ will be denoted by $\tilde{\nu}$, and for any sub-extension $L/K$ of $K_{\text{sep}}/K$, the normalized form of the valuation $\tilde{\nu}|_L$ on $L$ will be denoted by $\nu_L$. The completion of $K_{\text{nr}}$ with respect to the valuation $\nu_{K_{\text{nr}}}$ will be denoted by $\hat{K}$. For any separable extension $L/K$, we put $\tilde{L} := L\hat{K}$.

Let $G_K$ denote the absolute Galois group $\text{Gal}(K_{\text{sep}}/K)$. The topological generator of $\text{Gal}(K_{\text{nr}}/K)$, which is the Frobenius automorphism of $K$, is denoted by $\varphi_K = \varphi$ (if there is no risk of confusion). Any extension of the automorphism $\varphi : K_{\text{nr}} \to K_{\text{nr}}$ to $K_{\text{sep}}$ is called a Lubin–Tate splitting over $K$ and is again denoted by $\varphi$.

We further assume that the multiplicative group $\mu_p(K_{\text{sep}})$ of $p$th roots of unity in $K_{\text{sep}}$ satisfies

\begin{equation}
\mu_p(K_{\text{sep}}) \subseteq K.
\end{equation}

2.2. Local Artin reciprocity map. Let $G^\text{ab}_K$ denote the maximal abelian Hausdorff quotient group $G_K/G'_K$ of the topological group $G_K$, where $G'_K$ denotes the closure of the first commutator subgroup $[G_K, G_K]$ of $G_K$.

Recall that abelian local class field theory for the local field $K$ establishes a unique natural algebraic and topological isomorphism

$$\text{Art}_K : G^\text{ab}_K \sim \hat{K}^\times,$$

called the \emph{local Artin reciprocity map} of $K$, where the topological group $\hat{K}^\times$ denotes the pro-finite completion of the multiplicative group $K^\times$, satisfying certain properties. In particular, for an abelian extension $L/K$, and for every integer $0 \leq i \in \mathbb{Z}$ and real number $\nu \in (i - 1, i]$,

$$x \in U'_K \mathcal{N}_L \iff \text{Art}_{L/K}^{-1} \nu(x) \in \text{Gal}(L/K)^\nu,$$

where $x \in \hat{K}^\times$. Here, $\mathcal{N}_L$ denotes the closed subgroup of $\hat{K}^\times$ defined to be the intersection $\mathcal{N}_L = \bigcap_E N_{E/K}E^\times$, where $E$ runs over all finite extensions of $K$ inside $L$. 
In what follows, we shall briefly review the higher ramification subgroups in the upper numbering of the absolute Galois group $G_K$ of $K$.

2.3. A brief review of ramification theory. The main reference that we follow closely here is [Ik-Se-1].

For a finite separable extension $L/K$, and for any $\sigma \in \text{Hom}_K(L, K^{\text{sep}})$, introduce

$$i_{L/K}(\sigma) := \min_{x \in O_L} \nu_L(\sigma(x) - x),$$

put

$$\gamma_t := \# \{ \sigma \in \text{Hom}_K(L, K^{\text{sep}}) : i_{L/K}(\sigma) \geq t + 1 \},$$

for $-1 \leq t \in \mathbb{R}$, and define the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$, the Hasse–Herbrand transition function of the extension $L/K$, by

$$\varphi_{L/K}(u) := \begin{cases} \int_0^u \gamma_t \, dt, & 0 \leq u \in \mathbb{R}, \\ u, & -1 \leq u \leq 0. \end{cases}$$

It is well-known that $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a continuous, increasing, piecewise linear function, and it establishes a homeomorphism $\mathbb{R}_{\geq -1} \approx \mathbb{R}_{\geq -1}$. Let $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be its inverse.

Assume that $L$ is a finite Galois extension over $K$ with Galois group $\text{Gal}(L/K) =: G$. The normal subgroup $G_u$ of $G$ defined by

$$G_u = \{ \sigma \in G : i_{L/K}(\sigma) \geq u + 1 \}$$

for $-1 \leq u \in \mathbb{R}$ is called the $u$th ramification group of $G$ in the lower numbering, and has order $\gamma_u$. Note the inclusion $G_{u'} \subseteq G_u$ for every pair $-1 \leq u, u' \in \mathbb{R}$ satisfying $u \leq u'$. The family $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ induces a filtration on $G$, called the lower ramification filtration of $G$. A break in this filtration is defined to be any number $u \in \mathbb{R}_{\geq -1}$ satisfying $G_u \neq G_{u+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$. The function $\psi_{L/K} = \varphi_{L/K}^{-1} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ induces the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ on $G$ by setting

$$G^v := G_{\psi_{L/K}(v)},$$

or equivalently, by setting

$$G^{\varphi_{L/K}(u)} = G_u$$

for $-1 \leq v, u \in \mathbb{R}$; here $G^v$ is called the $v$th upper ramification group of $G$. A break in the upper filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of $G$ is defined to be any number $v \in \mathbb{R}_{\geq -1}$ satisfying $G^v \neq G^{v+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$.

Remark 2.1. We list the basic properties of lower and upper ramification filtrations on $G$. In what follows, $F/K$ denotes a sub-extension of $L/K$ and $H$ denotes the Galois group $\text{Gal}(L/F)$. 

\begin{enumerate}
\item For $-1 \leq u, v \in \mathbb{R}$, $G_u^v := G_u \cap G^v$.
\item For $-1 \leq u \in \mathbb{R}$, $G_u^\infty = G_u$.
\item For $-1 \leq u, v \in \mathbb{R}$, $G_u^v \subseteq G_u^w$ if $v \leq w$.
\item For $-1 \leq u, v \in \mathbb{R}$, $G_u^v \cap G_u^{v'} = G_u^{v''}$ if $v'' = \max(v, v')$.
\item For $-1 \leq u \in \mathbb{R}$, $G_u^0 = G_u$.
\end{enumerate}
(i) The lower numbering on $G$ passes to the subgroup $H$ of $G$ in the sense that 
\[ H_u = H \cap G_u \quad \text{for } -1 \leq u \in \mathbb{R}. \]

(ii) If $H \lhd G$, then the upper numbering on $G$ passes to the quotient $G/H$:
\[ (G/H)^v = G^v H/H \quad \text{for } -1 \leq v \in \mathbb{R}. \]

(iii) The Hasse–Herbrand function and its inverse satisfy the transitive law
\[ \varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F} \quad \text{and} \quad \psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}. \]

If $L/K$ is an infinite Galois extension with Galois group $\text{Gal}(L/K) = G$, which is a topological group under the respective Krull topology, define the upper ramification filtration $\{G^v\}_{v \in \mathbb{R} \geq -1}$ on $G$ by the projective limit
\[ (2.2) \quad G^v := \lim_{\longleftarrow} \text{Gal}(F/K)^v \]
over the transition morphisms $t_{F'}^{F}(v) : \text{Gal}(F'/K)^v \to \text{Gal}(F/K)^v$, which are essentially the restriction morphisms from $F'$ to $F$, defined naturally by the diagram
\[ (2.3) \begin{array}{c}
\text{Gal}(F/K)^v \\
\text{↓}
\end{array} \begin{array}{c}
t_{F}^{F'}(v)
\end{array} \begin{array}{c}
\text{Gal}(F'/K)^v
\end{array} \begin{array}{c}
\text{↓}
\end{array} \begin{array}{c}
\text{can.}
\end{array} \begin{array}{c}
\text{Gal}(F'/K)^v \text{Gal}(F'/F)/\text{Gal}(F'/F)
\end{array}
\]
induced from (ii), as $K \subseteq F \subseteq F' \subseteq L$ runs over all finite Galois extensions $F$ and $F'$ over $K$ inside $L$. The topological subgroup $G^v$ of $G$ is called the $v$th ramification group of $G$ in the upper numbering. Note the inclusion $G^{v'} \subseteq G^v$ for every pair $-1 \leq v, v' \in \mathbb{R}$ satisfying $v \leq v'$ via the commutativity of the square
\[ (2.4) \begin{array}{c}
\text{inc.}
\end{array} \begin{array}{c}
\text{Gal}(F/K)^v \\
\text{↓}
\end{array} \begin{array}{c}
t_{F}^{F'}(v)
\end{array} \begin{array}{c}
\text{inc.}
\end{array} \begin{array}{c}
\text{Gal}(F'/K)^v
\end{array} \begin{array}{c}
\text{↓}
\end{array} \begin{array}{c}
\text{inc.}
\end{array} \begin{array}{c}
\text{Gal}(F'/K)^{v'}
\end{array}
\]
for every chain $K \subseteq F \subseteq F' \subseteq L$ of finite Galois extensions $F$ and $F'$ over $K$ inside $L$. Observe that:

(iv) $G^{-1} = G$ and $G^0$ is the inertia subgroup of $G$.

(v) $\bigcap_{v \in \mathbb{R} \geq -1} G^v = \langle 1_G \rangle$. 

(vi) $G^v$ is a closed subgroup of $G$, with respect to the Krull topology, for $-1 \leq v \in \mathbb{R}$.

In this setting, a number $-1 \leq v \in \mathbb{R}$ is said to be a \textit{break in the upper ramification filtration} $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of $G$, if $v$ is a break in the upper filtration of some finite quotient $G/H$ for some $H \triangleleft G$. Let $\mathcal{B}_{L/K}$ denote the set of all numbers $v \in \mathbb{R}_{\geq -1}$ which occur as breaks in the upper ramification filtration of $G$. Then:

(vii) (Hasse–Arf theorem) $\mathcal{B}_{K^{ab}/K} \subseteq \mathbb{Z} \cap \mathbb{R}_{\geq -1}$.
(viii) $\mathcal{B}_{K^{sep}/K} \subseteq \mathbb{Q} \cap \mathbb{R}_{\geq -1}$.

2.4. APF-extensions. As in the previous section, let $\{G_K^v\}_{v \in \mathbb{R}_{\geq -1}}$ denote the upper ramification filtration of the absolute Galois group $G_K$ of $K$, and let $R^v$ denote the fixed field $(K^{sep})^{G_K^v}$ of the $v$th upper ramification subgroup $G_K^v$ of $G_K$ in $K^{sep}$ for $-1 \leq v \in \mathbb{R}$.

\textbf{Definition 2.2.} An extension $L/K$ is called an \textit{APF-extension} (APF is the shortening for “arithmétique profinie”) if one of the following equivalent conditions is satisfied:

(i) $G_K^v G_L$ is open in $G_K$ for every $-1 \leq v \in \mathbb{R}$,
(ii) $(G_K : G_K^v G_L) < \infty$ for every $-1 \leq v \in \mathbb{R}$,
(iii) $L \cap R^v$ is a finite extension over $K$ for every $-1 \leq v \in \mathbb{R}$.

Note that if $L/K$ is an APF-extension, then $[\kappa_L : \kappa_K] < \infty$.

Now, let $L/K$ be an APF-extension. Set $G_L^0 = G_L \cap G_K^0$, and define

$$
\psi_{L/K}(v) = \begin{cases} 
  \int_0^v (G_K^0 : G^0_L G^x_K) \, dx, & 0 \leq v \in \mathbb{R}, \\
  0, & -1 \leq v \leq 0.
\end{cases}
$$

Then the map $v \mapsto \psi_{L/K}(v)$ for $v \in \mathbb{R}_{\geq -1}$, which is well-defined for the APF-extension $L/K$, defines a continuous, strictly increasing and piecewise linear bijection $\psi_{L/K} : \mathbb{R}_{\geq -1} \rightarrow \mathbb{R}_{\geq -1}$.

We denote the inverse of $\psi_{L/K}$ by $\varphi_{L/K}$. Thus, if $L/K$ is a (not necessarily finite) Galois APF-extension, then we can define the higher ramification subgroups in the lower numbering $\text{Gal}(L/K)_u$ of $\text{Gal}(L/K)$, for $-1 \leq u \in \mathbb{R}$, by setting

$$
\text{Gal}(L/K)_u : = \text{Gal}(L/K)^{\varphi_{L/K}(u)}.
$$

\textbf{Remark 2.3.} Note that:

(i) In case $L/K$ is a finite separable extension, which is clearly an APF-extension by Definition 2.2, the function $\psi_{L/K} : \mathbb{R}_{\geq -1} \rightarrow \mathbb{R}_{\geq -1}$ coincides with the inverse of the Hasse–Herbrand transition function of $L/K$ introduced in the previous section.
(ii) If \( L/K \) is a finite separable extension and \( L'/L \) is an APF-extension, then \( L'/K \) is an APF-extension, and the transitivity rules for the functions \( \psi_{L'/K}, \varphi_{L'/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1} \) hold:
\[
\psi_{L'/K} = \psi_{L'/L} \circ \psi_{L/K}, \quad \varphi_{L'/K} = \varphi_{L/K} \circ \varphi_{L'/L}.
\]

3. Non-abelian local reciprocity map. In this section, we shall re-
view the theory developed in [Ik-Se-2, Ik-Se-3]. Fix a Lubin–Tate splitting \( \varphi \) over \( K \).

3.1. Generalized Fesenko reciprocity map. For an infinite APF-
Galois extension \( L/K \) with residue class degree \( [\kappa_L : \kappa_K] = d \) and with \( K \subset L \subset K_{\varphi^d} \), denote the field of norms corresponding to \( L/K \) by \( \mathbb{X}(L/K) \) and the completion of the maximal unramified extension \( \mathbb{X}(L/K)^{nr} \) of \( \mathbb{X}(L/K) \) by \( \tilde{\mathbb{X}}(L/K) \) (for details, [Fe-Vo], [Fo-Wi-1, Fo-Wi-2] and [Win]), and set \( L_0 = L \cap K^{nr} \). There exists a bijective 1-cocycle
\[
(3.1) \quad \Phi_{L/K}^{(\varphi)} : \text{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^\circ_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0},
\]
called the generalized Fesenko reciprocity map for the extension \( L/K \), de-
efined by the composition
\[
(3.2) \quad \begin{array}{ccc}
\text{Gal}(L/K) & \xrightarrow{\Phi_{L/K}^{(\varphi)}} & K^\times/N_{L_0/K}L_0^\times \times U^\circ_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \\
& \downarrow & \\
K^\times/N_{L_0/K}L_0^\times \times U^\circ_{\mathbb{X}(L/K)}/Y_{L/L_0} & \xrightarrow{(\text{id}_{K^\times/N_{L_0/K}L_0^\times \times U^\circ_{\tilde{\mathbb{X}}(L/K)}})} & \\
\end{array}
\]
Here,
\[
(3.3) \quad \Phi_{L/K}^{(\varphi)} : \text{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^\circ_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}
\]
is an injective 1-cocycle called, following [Ik-Se-2], the generalized arrow
defined for the extension \( L/K \), and defined by
\[
(3.4) \quad \Phi_{L/K}^{(\varphi)}(\sigma) = (\pi_{K/K}^m N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m} \sigma)),
\]
for every \( \sigma \in \text{Gal}(L/K) \), where \( 0 \leq m \in \mathbb{Z} \) is the integer satisfying \( \sigma|_{L_0} = \varphi^m|_{L_0} \in \text{Gal}(L_0/K) \) and \( \varphi^{-m} \sigma \in \text{Gal}(L/L_0) \), and for any \( \tau \in \text{Gal}(L/L_0) \), the value \( \phi_{L/L_0}^{(\varphi^d)}(\tau) \) of the arrow defined for the extension \( L/L_0 \) at \( \tau \) is de-
efined by [Fes-1, Fes-2, Fes-3] and [Ik-Se-1]. Namely, \( \phi_{L/L_0}^{(\varphi^d)}(\tau) = U_{\tau}.U_{\mathbb{X}(L/L_0)} \) provided that \( U_{\tau} \in U^\circ_{\mathbb{X}(L/K)} \), which is unique modulo \( U_{\mathbb{X}(L/L_0)} \), solves the equation \( U^{1-\varphi^d} = II_{\varphi^d;L/L_0}^{-1} \), where \( II_{\varphi^d;L/L_0} \) is the canonical prime element.
of the local field $\mathbb{X}(L/L_0)$ defined in Lemmas 1.2 and 1.3 of [Ik-Se-2]. For the
definition of the group $U_0^\circ_{\mathbb{X}(L/K)}$ and its subgroups $U_{\mathbb{X}(L/L_0)}$ and $Y_{L/L_0}$ satisfying
the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$ we refer the reader to [Fes-1, Fes-2, Fes-3]
and [Ik-Se-1]. In the commutative triangle (3.2), the arrow
\[(3.5)\]
$$c_{L/L_0} : U_0^\circ_{\mathbb{X}(L/K)} / U_{\mathbb{X}(L/K)} \to U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$$
is the canonical map defined by the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$. Recall
that (cf. [Fes-1, Fes-2, Fes-3] and [Ik-Se-1]) the composition $c_{L/L_0} \circ \phi^{(\varphi^d)}_{L/L_0} = \Phi_{L/L_0} : \text{Gal}(L/L_0) \to U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$ is the Fesenko reciprocity map for the
extension $L/L_0$. Thus, for $\sigma \in \text{Gal}(L/K)$, the value $\Phi_{L/K}^{(\varphi)}(\sigma)$ is defined by
\[(3.6)\]
$$\Phi_{L/K}^{(\varphi)}(\sigma) = (\pi^m_K N_{L_0/K} L_0^\times, \Phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m} \sigma)), \quad \text{where } 0 \leq m \in \mathbb{Z} \text{ satisfies } \sigma|_{L_0} = \varphi^m|_{L_0} \in \text{Gal}(L_0/K) \text{ and } \varphi^{-m} \sigma \in \text{Gal}(L/L_0).$$
Define a composition law $\ast$ on $\text{im}(\Phi_{L/K}^{(\varphi)})$ by
\[(3.7)\]
$$(\overline{a}, \overline{U}) \ast (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}), (\overline{b}, \overline{V}) (\Phi_{L/K}^{(\varphi)})^{-1}(\overline{a}, \overline{U})$$
for every $\overline{a} = a.N_{L_0/K} L_0^\times, \overline{b} = b.N_{L_0/K} L_0^\times \in K^\times / N_{L_0/K} L_0^\times$ with $a, b \in K^\times$ and $\overline{U} = U.U_{\mathbb{X}(L/K)}, \overline{V} = V.U_{\mathbb{X}(L/K)} \in U_0^\circ_{\mathbb{X}(L/K)} / U_{\mathbb{X}(L/K)}$ with $U, V \in U_0^\circ_{\mathbb{X}(L/K)}$; where the action of $\text{Gal}(L/K)$ on $\text{im}(\Phi_{L/K}^{(\varphi)})$ is defined by $(\overline{b}, \overline{V})^\sigma = (\overline{b}, \overline{V}^{\varphi^{-m}\sigma})$. Then $K^\times / N_{L_0/K} L_0^\times \times U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$ is a topological group
under $\ast$, and $\Phi_{L/K}^{(\varphi)}$ induces an isomorphism of topological groups
\[(3.8)\]
$$\Phi_{L/K}^{(\varphi)} : \text{Gal}(L/K) \sim \rightarrow \text{im}(\Phi_{L/K}^{(\varphi)}),$$
where the topological group structure on $\text{im}(\Phi_{L/K}^{(\varphi)})$ is defined with respect
to the binary operation $\ast$ defined by (3.7). Likewise, define a composition law, again denoted by $\ast$, on $K^\times / N_{L_0/K} L_0^\times \times U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$ by
\[(3.9)\]
$$(\overline{a}, \overline{U}) \ast (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}), (\overline{b}, \overline{V}) (\Phi_{L/K}^{(\varphi)})^{-1}(\overline{a}, \overline{U})$$
for every $\overline{a} = a.N_{L_0/K} L_0^\times, \overline{b} = b.N_{L_0/K} L_0^\times \in K^\times / N_{L_0/K} L_0^\times$ with $a, b \in K^\times$ and $\overline{U} = U.Y_{L/L_0}, \overline{V} = V.Y_{L/L_0} \in U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$ with $U, V \in U_0^\circ_{\mathbb{X}(L/K)}$; where the action of $\text{Gal}(L/K)$ on $K^\times / N_{L_0/K} L_0^\times \times U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$ is defined
by $(\overline{b}, \overline{V})^\sigma = (\overline{b}, \overline{V}^{\varphi^{-m}\sigma})$. Then $K^\times / N_{L_0/K} L_0^\times \times U_0^\circ_{\mathbb{X}(L/K)} / Y_{L/L_0}$ is a topolog-
chemical group under *, and $\Phi^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

\begin{equation}
\Phi^{(\varphi)}_{L/K} : \text{Gal}(L/K) \xrightarrow{\sim} K^\times/N_{L_0/K}L_0^\times \times U_{\chi(L/K)}^\circ/Y_{L/L_0},
\end{equation}

where the topological group structure on $K^\times/N_{L_0/K}L_0^\times \times U_{\chi(L/K)}^\circ/Y_{L/L_0}$ is defined with respect to the binary operation * defined by (3.9).

The mappings $\Phi^{(\varphi)}_{L/K}$ and $\Phi^{(\varphi)}_{L/K}$ have the following basic properties.

(i) For an infinite Galois sub-extension $M/K$ of $L/K$ such that $[\kappa_M : \kappa_K] = d'$ and $K \subset M \subset K_{d'}$ for some $d' \mid d$, the square

\begin{equation}
\begin{tikzcd}
\text{Gal}(L/K) \arrow[r, \Phi^{(\varphi)}_{L/K}] \arrow[d, \text{res}_M] & K^\times/N_{L_0/K}L_0^\times \times U_{\chi(L/K)}^\circ/Y_{L/L_0} \\
\text{Gal}(M/K) \arrow[r, \Phi^{(\varphi)}_{M/K}] & K^\times/N_{M_0/K}M_0^\times \times U_{\chi(M/K)}^\circ/Y_{M/M_0}
\end{tikzcd}
\end{equation}

is commutative, where the right vertical arrow is defined by

\begin{equation}
(e_{L_0/M_0, \tilde{N}_{L/M}^{\text{Coleman}}}^{\text{CFT}}) : (\bar{\alpha}, \bar{\xi}) \mapsto (e_{L_0/M_0}^{\text{CFT}}(\bar{\alpha}), \tilde{N}_{L/M}^{\text{Coleman}}(\bar{\xi}))
\end{equation}

for every $(\bar{\alpha}, \bar{\xi}) \in K^\times/N_{L_0/K}L_0^\times \times U_{\chi(L/K)}^\circ/Y_{L/L_0}$. Here,

$$\tilde{N}_{L/M}^{\text{Coleman}} : U_{\chi(L/K)}^\circ/Y_{L/L_0} \to U_{\chi(M/K)}^\circ/Y_{M/M_0}$$

is the Coleman norm map from $L$ to $M$ defined by equations (2.22) and (2.23) of [Ik-Se-2]. Likewise, the square

\begin{equation}
\begin{tikzcd}
\text{Gal}(L/K) \arrow[r, \Phi^{(\varphi)}_{L/K}] \arrow[d, \text{res}_M] & K^\times/N_{L_0/K}L_0^\times \times U_{\chi(L/K)}^\circ/Y_{L/L_0} \\
\text{Gal}(M/K) \arrow[r, \Phi^{(\varphi)}_{M/K}] & K^\times/N_{M_0/K}M_0^\times \times U_{\chi(M/K)}^\circ/Y_{M/M_0}
\end{tikzcd}
\end{equation}

is commutative, where the right vertical arrow is defined by

\begin{equation}
(e_{L_0/M_0, \tilde{N}_{L/M}^{\text{Coleman}}}^{\text{CFT}}) : (\bar{\alpha}, \bar{\xi}) \mapsto (e_{L_0/M_0}^{\text{CFT}}(\bar{\alpha}), \tilde{N}_{L/M}^{\text{Coleman}}(\bar{\xi}))
\end{equation}

for $(\bar{\alpha}, \bar{\xi}) \in K^\times/N_{L_0/K}L_0^\times \times U_{\chi(L/K)}^\circ/Y_{L/L_0}$. Here, $\tilde{N}_{L/M}^{\text{Coleman}} : U_{\chi(L/K)}^\circ/Y_{L/L_0} \to U_{\chi(M/K)}^\circ/Y_{M/M_0}$ is the Coleman norm map from $L$ to $M$ defined by Lemma 2.21 together with equations (2.47) and (2.48) of [Ik-Se-2]. Moreover, the arrow $e_{L_0/M_0}^{\text{CFT}} : K^\times/N_{L_0/K}L_0^\times \to K^\times/N_{M_0/K}M_0^\times$ appearing in both commutative diagrams is the natural inclusion defined via the existence theorem of local class field theory.
(ii) For each $0 \leq i \in \mathbb{R}$, introduce the subgroups \((U_n^o)^i\) of the field $\mathbb{X}(L/K)$ by \((U_n^o)^i = U_n^o \cap U^i_\mathbb{X}(L/K)\). For each $0 \leq n \in \mathbb{Z}$, as in equation (5.42) of \[I_k-Se-1\], let
\[
Q^n_{L/L_0} = c_{L/L_0}(U_n^o)^n U\mathbb{X}(L/K)/U\mathbb{X}(L/K) \cap \text{im}(\phi_{L/L_0}^{(\varphi)}),
\]
which is a subgroup of \((U_n^o)^n\). Here, the canonical homomorphism $c_{L/L_0}$ introduced in (3.5) is defined by equation (5.35) of \[I_k-Se-1\]. Now, the ramification theorem for the generalized arrow $\phi_{L/K}^{(\varphi)}$ yields, for $0 \leq n \in \mathbb{Z}$, the inclusion
\[
\phi_{L/K}^{(\varphi)}(\text{Gal}(L/K)\psi_{L/K}\psi_{L/K}(n) - \text{Gal}(L/K)\psi_{L/K}\psi_{L/K}(n+1))
\subseteq \langle 1_{K^x/NL_0/KL_0^x} \times ((U_n^o)^n U\mathbb{X}(L/K)/U\mathbb{X}(L/K)
- (U_n^o)^n U\mathbb{X}(L/K))/U\mathbb{X}(L/K) \rangle,
\]
and the ramification theorem for the generalized Fesenko reciprocity map $\Phi_{L/K}^{(\varphi)}$ gives, for $0 \leq n \in \mathbb{Z}$, the inclusion
\[
\Phi_{L/K}^{(\varphi)}(\text{Gal}(L/K)\psi_{L/K}\psi_{L/K}(n) - \text{Gal}(L/K)\psi_{L/K}\psi_{L/K}(n+1))
\subseteq \langle 1_{K^x/NL_0/KL_0^x} \times ((U_n^o)^n Y_{L/L_0}/Y_{L/L_0} - Q^n_{L/L_0}) \rangle,
\]
where, for $0 \leq u \in \mathbb{R}$, $\text{Gal}(L/K)_u$ denotes the $u$th ramification subgroup in the lower numbering of the Galois group $\text{Gal}(L/K)$ corresponding to the infinite APF-Galois extension $L/K$.

**Remark 3.1.** In fact, ramification theorems for $\phi_{L/K}^{(\varphi)}$ and $\Phi_{L/K}^{(\varphi)}$ stated in (3.16) and (3.17) can be simplified as follows. For $0 \leq n \in \mathbb{Z}$, as $\varphi_{L/K}(n) = \varphi_{L_0/K} \circ \varphi_{L/L_0}(n)$ and $L_0 = L \cap K^{nr}$, it follows that $\varphi_{L/K}(n) = \varphi_{L_0}(n)$. Therefore, (3.16) can be reformulated as
\[
\phi_{L/K}^{(\varphi)}(\text{Gal}(L/K)_n - \text{Gal}(L/K)_{n+1}) \subseteq \langle 1_{K^x/NL_0/KL_0^x} \times ((U_n^o)^n U\mathbb{X}(L/K)/U\mathbb{X}(L/K)
- (U_n^o)^n U\mathbb{X}(L/K))/U\mathbb{X}(L/K) \rangle,
\]
and (3.17) can be reformulated as
\[
\Phi_{L/K}^{(\varphi)}(\text{Gal}(L/K)_n - \text{Gal}(L/K)_{n+1})
\subseteq \langle 1_{K^x/NL_0/KL_0^x} \times ((U_n^o)^n Y_{L/L_0}/Y_{L/L_0} - Q^n_{L/L_0}) \rangle.
\]
Finally, the following remark is in order.

**Remark 3.2.** We do not need assumption (2.1) on the local field $K$ to define the generalized arrow $\phi_{L/K}^{(\varphi)}$ by (3.4). For details, cf. \[I_k-Se-2\].
3.2. Construction of the non-abelian local reciprocity map. For each $1 \leq d \in \mathbb{Z}$, let $K_{\varphi^d}$ denote the fixed field of $\varphi^d \in G_K$. Observe that $K_{\text{sep}} = K_{\text{nr}} \cap K_{\varphi^d}$ and $K_{\text{nr}}^d = K_{\text{nr}} \cap K_{\varphi^d}$, where $K_{\text{nr}}^d$ denotes the unique unramified extension over $K$ of degree $d$. Now, for each $1 \leq n, d \in \mathbb{Z}$, let $\Gamma_d^{(n)} := \Gamma_d^{(n)}(K, \varphi)$ be a Galois extension over $K$, which is the unique maximal $n$-abelian extension of $K_{\text{nr}}^d$ in $K_{\varphi^d}$. Note that

\[(3.20) \bigcup_{1 \leq d \in \mathbb{Z}} \Gamma_d^{(n)} = (K_{\text{nr}}^d)^{n-\text{ab}},\]

where $(K_{\text{nr}}^d)^{n-\text{ab}}$ denotes the “$n$-abelian closure” of $K_{\text{nr}}^d$ in $K_{\text{sep}}$. Thus, it also follows that

\[(3.21) \bigcup_{1 \leq n \in \mathbb{Z}} \bigcup_{1 \leq d \in \mathbb{Z}} \Gamma_d^{(n)} = K_{\text{sep}}.\]

Moreover, for each pair $(n, d)$ of positive integers, $\Gamma_d^{(n)}$ is an APF-extension over $K$.

Now, the absolute Galois group $G_K$ of the local field $K$ is the projective limit

\[G_K = \lim_{\leftarrow} \text{Gal}(\Gamma_d^{(n)}/K)\]

over the restriction morphisms

\[r_{(n,d)}^{(n,d)'} : \text{Gal}(\Gamma_d^{(n)}/K) \to \text{Gal}(\Gamma_{d'}^{(n)}/K)\]

for $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $n' \leq n$ and $d' | d$ (which is equivalent to $\Gamma_{d'}^{(n')} \subseteq \Gamma_d^{(n)}$). Note that, for each $1 \leq n, d \in \mathbb{Z}$, the APF-Galois extension $\Gamma_d^{(n)}$ over $K$ has the residue class degree $d$. Therefore, the generalized Fesenko theory developed in [Ik-Se-2] can be applied to the extensions of the form $\Gamma_d^{(n)}/K$, which would enable us to construct the generalized arrow $\Phi_{\Gamma_d^{(n)}}^{(\varphi)}$, and the generalized Fesenko reciprocity map $\Phi_{\Gamma_d^{(n)}/K}^{(\varphi)}$, for every pair $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. Then using property (i) for the collections $\{\phi_{\Gamma_d^{(n)}/K}^{(\varphi)}(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\}$ and $\{\Phi_{\Gamma_d^{(n)}/K}^{(\varphi)}(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\}$, and passing to the projective limits, we get the generalized arrow $\Phi_{\Gamma_d^{(n)}}^{(\varphi)}$ for the local field $K$ and the non-abelian local reciprocity map $\Phi_{\Gamma_d^{(n)}/K}^{(\varphi)}$ for the local field $K$ respectively.

To be more precise, we first introduce the following notation to simplify the discussion. In what follows, $L/K$ denotes an infinite APF-Galois extension such that $[\kappa_L : \kappa_K] = d$ and $K \subset L \subset K_{\varphi^d}$.

\(^{(1)}\) Recall that by an $n$-abelian extension over a field $F$, we mean a Galois extension $E/F$ whose Galois group $\text{Gal}(E/F)$ has a trivial $n$th commutator subgroup $\text{Gal}(E/F)^{(n)}$. 
NOTATION 3.3. For an infinite Galois sub-extension $M/K$ of $L/K$ such that $[\kappa_M : \kappa_K] = d'$ and $K \subset M \subset K_{\varphi^{d'}}$, provided that $d' \mid d$, we let

(i) $C_{L/M}^o$ denote the map $(c_{L_0/M_0}^{\text{CFT}}, \tilde{\alpha}_{L/M}^{\text{Coleman}})$ defined by (3.11) and (3.12),

(ii) $C_{L/M}$ denote the map $(c_{L_0/M_0}^{\text{CFT}}, \tilde{\alpha}_{L/M}^{\text{Coleman}})$ defined by (3.13) and (3.14).

Recall that $C_{L/M}^o : K^\times / N_{L_0/K} L_0^\times \times U_{\tilde{\chi}(L/K)}^o / U_{\tilde{\chi}(L/K)}$ and $C_{L/M} : K^\times / N_{L_0/K} L_0^\times \times U_{\tilde{\chi}(L/K)}^o / Y_{L_0/L_0} \rightarrow K^\times / N_{M_0/K} M_0^\times \times U_{\tilde{\chi}(M/K)}^o / Y_{M/M_0}$ are homomorphisms of the underlying abelian groups. Moreover, for the valued fields $L$ and $M$ as above, let $F/K$ be an infinite Galois sub-extension of $M/K$ satisfying $K \subset F \subset K_{\varphi^{d''}}$ with $[\kappa_F : \kappa_K] = d''$ where $d'' \mid d'$. If we set $F_0 = F \cap K^{nr}$, the following equalities hold:

(i) $C_{L/M}^o = \text{id}$ and $C_{L/M} = \text{id}$, if $L = M$.

(ii) $C_{L/F} = C_{M/F} \circ C_{L/M}^o$ and $C_{L/F} = C_{M/F} \circ C_{L/M}$.

It follows that the systems

\begin{equation}
(3.22) \quad \{ K^\times / N_{K_{d''}^{nr}} / K K_{d''}^{nr \times} \times U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}^o / U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)} ; C_{\Gamma_{d''}^{(n)}/\Gamma_{d''}^{(n')}}^o \}_{n' \leq n}^{d'' \mid d}
\end{equation}

and

\begin{equation}
(3.23) \quad \{ K^\times / N_{K_{d''}^{nr}} / K K_{d''}^{nr \times} \times U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}^o / Y_{\Gamma_{d''}^{(n)}/K_{d''}^{nr}} ; C_{\Gamma_{d''}^{(n)}/\Gamma_{d''}^{(n')}}^o \}_{n' \leq n}^{d'' \mid d}
\end{equation}

are projective. Let

\begin{equation}
(3.24) \quad \nabla_{K}^{(\varphi),o} = \nabla_{K}^{o} = \lim_{(n,d)} K^\times / N_{K_{d''}^{nr}} / K K_{d''}^{nr \times} \times U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}^o / U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}
\end{equation}

\begin{equation}
= \hat{\Z} \times \lim_{(n,d)} U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}^o / U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}
\end{equation}

and

\begin{equation}
(3.25) \quad \nabla_{K}^{(\varphi)} = \nabla_{K} = \lim_{(n,d)} K^\times / N_{K_{d''}^{nr}} / K K_{d''}^{nr \times} \times U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}^o / Y_{\Gamma_{d''}^{(n)}/K_{d''}^{nr}}
\end{equation}

\begin{equation}
= \hat{\Z} \times \lim_{(n,d)} U_{\tilde{\chi}(\Gamma_{d''}^{(n)}/K)}^o / Y_{\Gamma_{d''}^{(n)}/K_{d''}^{nr}}
\end{equation}

be the projective limits of the systems (3.22) and (3.23) respectively. The limits $\nabla_{K}^{(\varphi),o}$ and $\nabla_{K}^{(\varphi)}$, or $\nabla_{K}^{o}$ and $\nabla_{K}$ respectively if there is no risk of confusion, depend on the choice of a Lubin–Tate splitting $\varphi$ over $K$. 

Ramification theory
Note that $\nabla^0_K$ and $\nabla_K$ have natural topological $G_K$-module structures, where the $G_K$-action on $\nabla^0_K$ and on $\nabla_K$ is defined by

$$((\bar{a}_{d,n}, \bar{U}_{d,n}))_{d,n}^\sigma = ((\bar{a}_{d,n}, \bar{U}_{d,n})^\sigma \mid_{I_d^{(n)}})_{d,n}$$

for every coherent sequence $((\bar{a}_{d,n}, \bar{U}_{d,n}))_{d,n}$ from $\nabla^0_K$ or from $\nabla_K$, and for every $\sigma \in G_K = \lim_{(n,d)} \text{Gal}(\Gamma_d^{(n)}/K)$.

For any two pairs $(n, d)$ and $(n', d')$ satisfying $n' \leq n$ and $d' \mid d$, the square

$$
\begin{array}{ccc}
U^\phi \overline{\mathbb{X}}(\Gamma_d^{(n)}/K) & \to & U^\phi \overline{\mathbb{X}}(\Gamma_d^{(n)}/K) \\
\downarrow \scriptstyle{\tilde{K}}^{\text{Coleman}}_{\Gamma_d^{(n)}/I_d^{(n')}} & & \downarrow \scriptstyle{\tilde{K}}^{\text{Coleman}}_{\Gamma_d^{(n)}/I_d^{(n')}} \\
Y_{I_d^{(n)}/K}^{n_d} & \to & Y_{I_d^{(n)}/K}^{n_d}
\end{array}
$$

is commutative. Therefore, the topological $G_K$-modules $\nabla^0_K$ and $\nabla_K$ are related to each other by a topological $G_K$-module homomorphism

$$c_K := \lim_{(n,d)} (\text{id}_{K^\times/K} \otimes_{K_{d,n}^\times} C_{\Gamma_d^{(n)}} \otimes_{K_{d,n}^\times} c_{\Gamma_d^{(n)}}) : \nabla^0_K \to \nabla_K$$

defined by the commutativity of the diagram (3.27).

Therefore, there exists an injective map

$$\phi^{(\varphi)}_K = \lim_{(n,d)} \phi^{(\varphi)}_{I_d^{(n)}} : G_K \to \nabla^0_K$$

defined by

$$\phi^{(\varphi)}_K ((\sigma_{d,n})_{d,n}) = (\phi^{(\varphi)}_{I_d^{(n)}} (\sigma_{d,n}))_{d,n}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \lim_{(n,d)} \text{Gal}(\Gamma_d^{(n)}/K) = G_K$, and a bijective map

$$\Phi^{(\varphi)}_K = \lim_{(n,d)} \Phi^{(\varphi)}_{I_d^{(n)}} : G_K \to \nabla_K$$

defined by

$$\Phi^{(\varphi)}_K ((\sigma_{d,n})_{d,n}) = (\Phi^{(\varphi)}_{I_d^{(n)}} (\sigma_{d,n}))_{d,n}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \lim_{(n,d)} \text{Gal}(\Gamma_d^{(n)}/K) = G_K$. Moreover, the injective mapping $\phi^{(\varphi)}_K : G_K \to \nabla^0_K$ is a 1-cocycle, that is, for $\sigma, \tau \in G_K$ with respective coherent sequences $(\sigma_{d,n})_{d,n}, (\tau_{d,n})_{d,n} \in \lim_{(n,d)} \text{Gal}(\Gamma_d^{(n)}/K)$,

$$\phi^{(\varphi)}_K (\sigma \tau) = \phi^{(\varphi)}_K (\sigma) \phi^{(\varphi)}_K (\tau)^\sigma.$$
Also the bijective mapping $\Phi_K^{(\varphi)} : G_K \to \nabla_K$ is a 1-cocycle, i.e., for $\sigma, \tau \in G_K$ with respective coherent sequences $(\sigma_{d,n})_{d,n}, (\tau_{d,n})_{d,n} \in \varprojlim \text{Gal}(I_d^{(n)}/K)$,

$$\Phi_K^{(\varphi)}(\sigma\tau) = \Phi_K^{(\varphi)}(\sigma)\Phi_K^{(\varphi)}(\tau) \cdot \sigma.$$

**Definition 3.4.** The injective 1-cocycle $\Phi_K^{(\varphi)} : G_K \to \nabla_K^0$ is called the generalized arrow for $K$, and the bijective 1-cocycle $\Phi_K^{(\varphi)} : G_K \to \nabla_K$ is called the non-abelian local reciprocity map of $K$.

The 1-cocycles $\Phi_K^{(\varphi)}$ and $\Phi_K^{(\varphi)}$ are related to each other by

$$\Phi_K^{(\varphi)} = c_K \circ \Phi_K^{(\varphi)}.$$  

**4. Ramification theory.** Now, by Theorems 2.7 and 2.20 of [Ik-Se-2] (cf. also Remark 3.1 in Section 3), the ramification theorems for the generalized arrow $\Phi_{I_d^{(n)}/K}$ and for the generalized Fesenko reciprocity map $\Phi_{I_d^{(n)}/K}^{(\varphi)}$ give, for $0 \leq w \in \mathbb{Z}$, the inclusions

$$\Phi_{I_d^{(n)}/K}^{(\varphi)}((\text{Gal}(I_d^{(n)}/K)_w - \text{Gal}(I_d^{(n)}/K)_{w+1})$$

$$\subseteq \langle 1_{K^{\times}/N_{K_d^{nr}/K}K_d^{nr\times}} \times ((U_{\kappa(I_d^{(n)}/K)})^wU_X(I_d^{(n)}/K)/U_X(I_d^{(n)}/K),$$

and

$$\Phi_{I_d^{(n)}/K}^{(\varphi)}((\text{Gal}(I_d^{(n)}/K)_w - \text{Gal}(I_d^{(n)}/K)_{w+1})$$

$$\subseteq \langle 1_{K^{\times}/N_{K_d^{nr}/K}K_d^{nr\times}} \times ((U_{\kappa(I_d^{(n)}/K)})^wY_{I_d^{(n)}/K_d^{nr}/Y_{I_d^{(n)}/K_d^{nr}}} - Q_{I_d^{(n)}/K_d^{nr}}^{w+1},$$

where $\text{Gal}(I_d^{(n)}/K)_w$ denotes the $w$th higher ramification subgroup in the lower numbering of the Galois group $\text{Gal}(I_d^{(n)}/K)$ corresponding to the infinite APF-Galois extension $I_d^{(n)}/K$.

The aim of this section is to state and prove ramification theorems for the generalized arrow $\Phi_K^{(\varphi)} : G_K \to \nabla_K^0$ and for the non-abelian local reciprocity map $\Phi_K^{(\varphi)} : G_K \to \nabla_K$.

**4.1. Higher ramification subgroups of $G_K$ in the upper numbering.** To simplify the discussion, we introduce the following notation.

**Notation 4.1.** For every $1 \leq d, n \in \mathbb{Z}$, the Galois group $\text{Gal}(I_d^{(n)}/K)$ is denoted by $G(d,n)$. Moreover, for any $-1 \leq w \in \mathbb{R}$, $G(d,n)^w$ denotes the $w$th ramification subgroup of $G(d,n)$ in the upper numbering.
PROPOSITION 4.2. For \((n', d'), (n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\) satisfying \(n' \leq n\) and \(d' \mid d\), and for \(0 \leq w \in \mathbb{Z}\),
\begin{equation}
\psi_{r_{d'}^{(n')}/K_{d'}^{nr}}(w) \leq \psi_{r_{d}^{(n)}/K_{d}^{nr}}(w).
\end{equation}

Proof. As \(r_{d'}^{(n')}/d\) and \(r_{d}^{(n)}/d\) are APF-extensions over the field \(K\), for every \(-1 < x \in \mathbb{R}\), setting \(G_{r_{d'}^{(n')}}^{0} = G_{K}^{r_{d'}^{0}} \cap G_{r_{d'}^{(n')}}^{0}\) and \(G_{r_{d}^{(n)}}^{0} = G_{K}^{r_{d}^{0}} \cap G_{r_{d}^{(n)}}^{0}\), we have \((G_{K}^{0} : G_{K}^{x} G_{r_{d'}^{0} / r_{d'}^{(n')}}^{0}) < \infty\) and \((G_{K}^{0} : G_{K}^{x} G_{r_{d}^{0} / r_{d}^{(n)}}^{0}) < \infty\) (cf. [Fo-Wi-1, Fo-Wi-2].

Now, if \(n' \leq n\) and \(d' \mid d\), then \(r_{d'}^{(n')} \subseteq r_{d}^{(n)}\). Therefore,
\begin{equation}
(G_{K}^{0} : G_{K}^{x} G_{r_{d'}^{0} / r_{d'}^{(n')}}^{0}) \leq (G_{K}^{0} : G_{K}^{x} G_{r_{d}^{0} / r_{d}^{(n)}}^{0}) \leq \infty,
\end{equation}
as \(G_{r_{d}^{(n)}} \subseteq G_{r_{d'}^{(n')}}\). Hence, for \(0 \leq w \in \mathbb{Z}\),
\[
\psi_{r_{d'}^{(n')}/K}^{w}(w) = \int_{0}^{w} (G_{K}^{0} : G_{K}^{x} G_{r_{d'}^{0} / r_{d'}^{(n')}}^{0}) dx \leq \int_{0}^{w} (G_{K}^{0} : G_{K}^{x} G_{r_{d}^{0} / r_{d}^{(n)}}^{0}) dx = \psi_{r_{d}^{(n)}/K}^{w}(w).
\]
Now, the desired inequality follows, because
\[
\psi_{r_{d'}^{(n')}/K}^{w}(w) = \psi_{r_{d'}^{(n')}/K_{d'}^{nr}}^{w} \circ \psi_{K_{d'}^{nr}/K}^{w}(w) = \psi_{r_{d'}^{(n')}/K_{d'}^{nr}}^{w}(w)
\]
and likewise
\[
\psi_{r_{d}^{(n)}/K}^{w}(w) = \psi_{r_{d}^{(n)}/K_{d}^{nr}}^{w} \circ \psi_{K_{d}^{nr}/K}^{w}(w) = \psi_{r_{d}^{(n)}/K_{d}^{nr}}^{w}(w).
\]

REMARK 4.3. Note that Proposition 4.2 is more generally true in the following setting. Let \(L\) be an infinite APF-Galois extension over \(K\) satisfying \(K \subseteq L \subseteq \mathbb{K}_{\varphi^{d'}}\) with \([\varphi_{L} : \varphi_{K}] = d\), and \(M/K\) be an infinite Galois sub-extension of \(L/K\) satisfying \(K \subseteq M \subseteq \mathbb{K}_{\varphi^{d'}}\) with \([\varphi_{M} : \varphi_{K}] = d'\), where \(d' \mid d\). Then, for \(0 \leq w \in \mathbb{Z}\),
\[
\psi_{M/M_{0}}^{w}(w) \leq \psi_{L/L_{0}}^{w}(w),
\]
where \(L_{0} \subseteq L \cap K_{nr}\) and \(M_{0} = M \cap K_{nr}\). The proof follows the same lines.

It is well-known that, for a fixed \(-1 \leq w \in \mathbb{R}\), the projective limit
\begin{equation}
G_{K}^{w} := \lim_{\longrightarrow} G(d, n)^{w}
\end{equation}
over the restriction morphisms
\[
r_{(n,d)}^{(n',d')} : G(d, n)^{w} \to G(d', n')^{w}
\]
for \((n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\) satisfying \(n' \leq n\) and \(d' \mid d\) defines a subgroup \(G_{K}^{w}\) of the absolute Galois group \(G_{K}\), and we have the following definition.

DEFINITION 4.4. For \(-1 \leq w \in \mathbb{R}\), the group \(G_{K}^{w}\) is called the \(w\)th higher ramification subgroup of \(G_{K}\) in the upper numbering.
However, it turns out that we need a finer upper ramification “filtration” of $G_K$. Let $w := (w_{(n,d)})$ be a net in $\mathbb{R}_{\geq -1}$ always assumed to be indexed over the directed set $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, where $(n', d') \preceq (n, d)$ if $n' \leq n$ and $d' \mid d$ for $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. Furthermore, assume that the net $w$ in $\mathbb{R}_{\geq -1}$ is increasing, that is, $w_{(n',d')} \leq w_{(n,d)}$ if $(n', d') \preceq (n, d)$. In case $w = (w_{(n,d)})$ in $\mathbb{R}_{\geq -1}$ is constant, that is, $w_{(n,d)} = c$ for every $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, the net $w$ will be simply denoted by $c$.

Note that, for an increasing net $w$ in $\mathbb{R}_{\geq -1}$, the projective limit

$$G_w^K := \lim_{(n,d)} G(d, n)^{w_{(n,d)}}$$

over the restriction morphisms

$$r^{(n,d)}_{(n',d')} : G(d, n)^{w_{(n,d)}} \to G(d', n')^{w_{(n',d')}} \hookrightarrow G(d', n')^{w_{(n',d')}}$$

for $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $n' \leq n$ and $d' \mid d$ defines a subgroup $G_w^K$ of the absolute Galois group $G_K$, and we have the following definition.

**Definition 4.5.** For an increasing net $w$ in $\mathbb{R}_{\geq -1}$, the group $G_w^K$ is called the $w$th higher ramification subgroup of $G_K$ in the upper numbering.

**Definition 4.6.** Let $w = (w_{(n,d)})$ be an increasing net in $\mathbb{R}_{\geq -1}$. The net $w'$ in $\mathbb{R}_{\geq -1}$ defined by

$$w'_{(n,d)} = \varphi_{r^{(n)/K_d^{nr}}} (\psi_{r^{(n)/K_d^{nr}}}(w_{(n,d)}) + 1),$$

for every pair $(n, d)$, which is clearly an increasing net in $\mathbb{R}_{\geq -1}$, is called the successor of $w$.

Note that, for any increasing net $w$ in $\mathbb{R}_{\geq -1}$, we have the inclusion

$$G_w'^K \subseteq G_w^K,$$

because $G(d, n)^{\psi_{r^{(n)/K_d^{nr}}}(w_{(n,d)})+1} \subseteq G(d, n)^{\psi_{r^{(n)/K_d^{nr}}}(w_{(n,d)})}$ for every pair $(n, d)$. The proof of the following lemma is clear.

**Lemma 4.7.** For any increasing net $w$ in $\mathbb{R}_{\geq -1}$ and for $\sigma = (\sigma_{d,n})_{d,n} \in \lim_{(n,d)} G(d, n)^{w_{(n,d)}} = G_w^K$, the following two conditions are equivalent.

(i) $\sigma \in G_w'^K - G_w^K$.

(ii) $\sigma_{d,n} \in G(d, n)^{\psi_{r^{(n)/K_d^{nr}}}(w_{(n,d)})} - G(d, n)^{\psi_{r^{(n)/K_d^{nr}}}(w_{(n,d)})+1}$ for $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

**4.2. The groups $\nabla_w^{o,K}$ and $\nabla_w^K$ for an increasing net $w$ in $\mathbb{R}_{\geq -1}$.** The following proposition is central to what follows.

**Proposition 4.8.** Let $L$ be an infinite APF-Galois extension over $K$ satisfying $K \subseteq L \subseteq K_{\varphi^d}$ with $[\kappa_L : \kappa_K] = d$, and $M/K$ be an infinite Galois
sub-extension of $L/K$ satisfying $K \subset M \subset K_{\varphi^d}$ with $[\kappa_M : \kappa_K] = d'$, where $d' \mid d$. Then

\begin{equation}
(4.8) \quad \tilde{N}_{L/M} \circ \langle \varphi \rangle_{L/M} \left( (U^\infty_{\tilde{X}(L/K)})^w \right) \subseteq \left( U^\infty_{\tilde{X}(M/K)} \right)^w
\end{equation}

for every $0 \leq w \in \mathbb{Z}$.

**Proof.** For $0 \leq w \in \mathbb{Z}$, let $\alpha = (\alpha_{E_i})_{0 \leq i \in \mathbb{Z}} \in (U^\infty_{\tilde{X}(L/K)})^w$. That is, the norm coherent sequence $\alpha = (\alpha_{E_i})_{0 \leq i \in \mathbb{Z}} \in U^\infty_{\tilde{X}(L/K)}$ satisfies

\[
\nu_{\tilde{X}(L/K)}((\alpha_{E_i})_{0 \leq i \in \mathbb{Z}} - 1) = \nu_{\tilde{X}(L/K)}(\alpha_{E_i} - 1) = w,
\]

where the equality follows from the definition of addition on $\tilde{X}(L/K)$ and the valuation $\nu_{\tilde{X}(L/K)}$ on $\tilde{X}(L/K)$. Thus, by the definition of the mapping $\tilde{N}_{L/M} \circ \langle \varphi \rangle_{L/M} : \tilde{X}(L/K) \times \tilde{Y}(M/K) \to \tilde{X}(M/K) \times$, it follows that

\[
\nu_{\tilde{X}(M/K)} \left( \tilde{N}_{L/M} \circ \langle \varphi \rangle_{L/M} \left( (\alpha_{E_i})_{0 \leq i \in \mathbb{Z}} \right) - 1 \right) = \nu_{\tilde{X}(M/K)} \left( \alpha_{E_i}^\varphi - 1 \right) = \nu_{\tilde{X}(M/K)} \left( \sum_{0 \leq i \leq f(M/K)^2} \alpha_i^\ell \right) \geq w,
\]

which shows that $\tilde{N}_{L/M} \circ \langle \varphi \rangle_{L/M} \left( (\alpha_{E_i})_{0 \leq i \in \mathbb{Z}} \right) \in U^w_{\tilde{X}(M/K)}$. Combining this with the property (ii) of equation (2.21) in [Ik-Se-2] yields the assertion. 

**Notation 4.9.** Let $L$ be an infinite APF-Galois extension over $K$ satisfying $K \subset L \subset K_{\varphi^d}$ with $[\kappa_L : \kappa_K] = d$. For $0 \leq w \in \mathbb{R}$, let

\begin{equation}
(4.9) \quad (\nabla_{L/K}^w)^w = \nabla_{L/K}^{o,w} = K^\times/N_{L_0/L}K_0^\times \times (U^\infty_{\tilde{X}(L/K)})^{wU_{\tilde{X}(L/K)}/U_{\tilde{X}(L/K)}};
\end{equation}

\begin{equation}
(4.10) \quad (\nabla_{L/K}^\varphi)^w = \nabla_{L/K}^{w} = K^\times/N_{L_0/L}K_0^\times \times (U^\infty_{\tilde{X}(L/K)})^{wY_{L/L_0}/Y_{L_0}}.
\end{equation}

Therefore, by Remark 4.3 for the local fields $L$ and $M$ as in Proposition 4.8 and for $0 \leq w_{L/K}, w_{M/K} \in \mathbb{R}$ satisfying $w_{M/K} \leq w_{L/K}$, the map $C_{L/M}^o$ introduced in Notation 3.3(i) restricts to

\[ C_{L/M}^o : \nabla_{L/K}^{o,w_{L/L_0}^w} (w_{L/K}) \to \nabla_{M/K}^{o,w_{M/M_0}^w} (w_{M/K}), \]

and the map $C_{L/M}$ introduced in Notation 3.3(ii) restricts to

\[ C_{L/M} : \nabla_{L/K}^{w_{L/L_0}^w} (w_{L/K}) \to \nabla_{M/K}^{w_{M/M_0}^w} (w_{M/K}). \]
Thus, the following corollary follows directly.

**Corollary 4.10.** For an increasing net $\underline{w} = (w_{(n,d)})$ in $\mathbb{R}_{\geq 0}$, the systems

$$\{\nabla_{I_d^{(n)}/K}(w_{(n,d)})^{o,\psi} \}_{n' \leq n} \quad \text{and} \quad \{\nabla_{I_d^{(n)}/K^{nr}}(w_{(n,d)})^{\psi} \}_{n' \leq n}$$

are projective.

**Proof.** Follows from the projectivity of the systems (3.22) and (3.23), and from Proposition 4.8 combined with Proposition 4.2.

For any increasing net $\underline{w}$ in $\mathbb{R}_{\geq 0}$, let

$$\nabla_{K}^{(φ,o)} w := \nabla_{K}^{o,w} = \lim_{(n,d)} \nabla_{I_d^{(n)}/K}(w_{(n,d)})^{o,\psi}$$

$$= \widehat{\mathbb{Z}} \times \lim_{(n,d)} (U_{\overline{X}(I_d^{(n)}/K)})^{\psi} I_d^{(n)}/K^{nr}(w_{(n,d)}) U_{X(I_d^{(n)}/K)}/U_{X(I_d^{(n)}/K)}$$

and

$$\nabla_{K}^{(φ)} w := \nabla_{K}^{w} = \lim_{(n,d)} \nabla_{I_d^{(n)}/K}(w_{(n,d)})^{\psi}$$

$$= \widehat{\mathbb{Z}} \times \lim_{(n,d)} (U_{\overline{X}(I_d^{(n)}/K)})^{\psi} I_d^{(n)}/K^{nr}(w_{(n,d)}) Y_{I_d^{(n)}/K^{nr}}/Y_{I_d^{(n)}/K^{nr}}$$

be the projective limits of the systems (4.11) and (4.12) respectively. These limits $\nabla_{K}^{(φ,o)} w$ and $\nabla_{K}^{(φ)} w$, or $\nabla_{K}^{o,w}$ and $\nabla_{K}^{w}$ if there is no risk of confusion, depend on the choice of a Lubin–Tate splitting $φ$ over $K$.

**Lemma 4.11.** For $(n, d), (n', d')$ satisfying $n' \leq n, d' | d$, and for $0 \leq w_{(n,d)}, w_{(n',d')} \in \mathbb{R}$ satisfying $w_{(n',d')}$ $\leq w_{(n,d)}$ and $0 \leq \psi_{I_d^{(n)}/K}(w_{(n,d)})$, $\psi_{I_d^{(n')}/K}(w_{(n',d')}) \in \mathbb{Z}$, the squares

$$G(d, n) \xrightarrow{\phi_{I_d^{(n)}/K}^{(φ)}} A_{(n,d)}^{(U)}$$

$$\xrightarrow{r_{(n,d)}^{(n,d)}} G(d', n') \xrightarrow{\phi_{I_d^{(n')}/K}^{(φ)}} A_{(n',d')}^{(U)}$$

$$\xrightarrow{c_{I_d^{(n)}/I_d^{(n')}}^{o}} A_{(n,d)}$$

$$\xrightarrow{c_{I_d^{(n')}/I_d^{(n')}}^{o}} A_{(n',d')}$$
and

\[ G(d, n)^{w(n, d)} \xrightarrow{\phi^{(\psi)}_{r_d^n/K}} A^{(Y)}_{(n, d)} \]

\[ r_{(n, d)^{w(n', d')}} \]

(4.16)

\[ G(d, n)^{w(n', d')} \xrightarrow{\phi^{(\psi)}_{r_{d'}^{(n')}}} A^{(Y)}_{(n', d')} \]

are commutative, where for \((n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\) and \(0 \leq w(n, d) \in \mathbb{R}\) satisfying \(0 \leq \psi_{r_d^n/K}(w(n, d)) \in \mathbb{Z}\),

\[ A^{(U)}_{(n, d)} := \langle 1_{K^{x/N_K^{nr}}/K^{nr}} \rangle \times (U_{\mathfrak{X}(I_d^{(n)}/K)})^{\psi_{r_d^n/K}(w(n, d))} U_{\mathfrak{X}(I_d^{(n)}/K)}/U_{\mathfrak{X}(I_d^{(n)}/K)}, \]

\[ A^{(Y)}_{(n, d)} := \langle 1_{K^{x/N_K^{nr}}/K^{nr}} \rangle \times (U_{\mathfrak{X}(I_d^{(n)}/K)})^{\psi_{r_d^n/K}(w(n, d))} \gamma_{I_d^{(n)}/K^{nr}} / \gamma_{I_d^{(n)}/K^{nr}}. \]

Proof. Follows from Proposition 4.2 and the basic property (iii) of ramification theory of \(\phi^{(\psi)}_{L/K}\) and \(\phi^{(\psi)}_{L/K}\) together with the basic property (i) of \(\phi^{(\psi)}_{L/K}\) and \(\phi^{(\psi)}_{L/K}\) stated in Section 3.\]

For any increasing net \(w = (w(n, d))\) in \(\mathbb{R}_{\geq 0}\), let \(1_{\nabla_K^{o,w}}\) denote the kernel of the projection \(P_1 : \nabla_K^{o,w} \to \hat{\mathbb{Z}}\), and \(1_{\gamma_K^{w}}\) denote the kernel of the projection \(P_1 : \nabla_K^{w} \to \hat{\mathbb{Z}}\). An immediate consequence of Lemma 4.11 is

**Corollary 4.12.** Let \(w\) be any increasing net in \(\mathbb{R}_{\geq 0}\) and \(\sigma \in G_K^{w}\). Then:

(i) \(\phi^{(\psi)}_{K}(\sigma) \in 1_{\nabla_K^{w}}\).

(ii) \(\phi^{(\psi)}_{K}(\sigma) \in 1_{\gamma_K^{w}}\).

Now, for an increasing net \(w = (w(n, d))\) in \(\mathbb{R}_{\geq 0}\), introduce

\[ Q_K^{w} = c_K(1_{\nabla_K^{o,w}} \cap \text{im}(\phi^{(\psi)}_{K})), \]

where \(c_K : \nabla_K^{o} \to \nabla_K\) is the canonical map defined by (3.28). Note that \(c_K(1_{\nabla_K^{o,w}}) = 1_{\gamma_K^{w}}\) by the commutativity of the square (3.27) and by Propositions 4.2 and 4.8.

**Lemma 4.13.** For an increasing net \(w = (w(n, d))\) in \(\mathbb{R}_{\geq 0}\),

\[ \left\{ Q_{r_d^{(n)}/K^{nr}}^{(w(n, d))} : \mathcal{N}_{r_d^{(n)}/r_d^{(n')}}^{(n')} \right\}_{n' \leq n \atop d' | d} \]

is a projective system and its projective limit is

\[ Q_K^{w} = \langle 1_{\mathbb{Z}} \rangle \times \lim_{(n, d)} Q_{r_d^{(n)}/K^{nr}}^{(w(n, d))}. \]
Proof. The projectivity of (4.18) follows from the projectivity of the system \{\lim(\phi_{F_d}^{(\varphi)}/K), \mathcal{C}^0_{F_d} / F_{d'}^{(\varphi')}, n' \leq n \}
combined with Proposition 4.8 and (3.27). Moreover, the equality (4.19) follows from (3.28) and (3.27). \[\blacksquare\]

In the lemma below, whose proof is clear, \(w'\) denotes the successor of \(w\).

**Lemma 4.14.** (i) For any increasing net \(w = (w_{(n,d)})\) in \(\mathbb{R}_{\geq -1}\) and for an element \(u = (u_{(d,n)}, d,n) = ((1_{K^\times}/N_{K^\mathrm{nr}_d/K^\mathrm{nr}_d} \times, U_{d,n})\) of

\[\nabla_{K}^{w} = \langle 1_{Z} \rangle \times \lim_{(n,d)} (U_{d,n}^\circ / K^{(n)}/K) \psi_{F_d}^{(\varphi)} / K^{nr}_{(n,d)} U_{\chi(I_d^{(n)}/K)} / U_{\chi(I_d^{(n)}/K)},\]

we have: \(u \in 1\nabla_{K}^{w} - 1\nabla_{K}^{w'}\) if and only if

\[u_{(d,n)} = (1_{K^\times}/N_{K^\mathrm{nr}_d/K^\mathrm{nr}_d} \times, U_{d,n}) \in \langle 1_{K^\times}/N_{K^\mathrm{nr}_d/K^\mathrm{nr}_d} \times, U_{d,n} \rangle\]

\[\times ((U_{d,n}^\circ / K^{(n)}/K) \psi_{F_d}^{(\varphi)} / K^{nr}_{(n,d)} U_{\chi(I_d^{(n)}/K)} / U_{\chi(I_d^{(n)}/K)} - (U_{d,n}^\circ / K^{(n)}/K) \psi_{F_d}^{(\varphi)} / K^{nr}_{(n,d)+1} U_{\chi(I_d^{(n)}/K)} / U_{\chi(I_d^{(n)}/K)}),\]

for some \((n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\).

(ii) For any increasing net \(w = (w_{(n,d)})\) in \(\mathbb{R}_{\geq -1}\) and for an element \(u = (u_{(d,n)}, d,n) = ((1_{K^\times}/N_{K^\mathrm{nr}_d/K^\mathrm{nr}_d} \times, U_{d,n})\) of

\[\nabla_{K}^{w} = \langle 1_{Z} \rangle \times \lim_{(n,d)} (U_{d,n}^\circ / K^{(n)}/K) \psi_{F_d}^{(\varphi)} / K^{nr}_{(n,d)} U_{\chi(I_d^{(n)}/K)} / U_{\chi(I_d^{(n)}/K)},\]

we have: \(u \in 1\nabla_{K}^{w} - Q_{K}^{w'}\) if and only if

\[u_{(d,n)} = (1_{K^\times}/N_{K^\mathrm{nr}_d/K^\mathrm{nr}_d} \times, U_{d,n}) \in \langle 1_{K^\times}/N_{K^\mathrm{nr}_d/K^\mathrm{nr}_d} \times, U_{d,n} \rangle\]

\[\times ((U_{d,n}^\circ / K^{(n)}/K) \psi_{F_d}^{(\varphi)} / K^{nr}_{(n,d)} U_{\chi(I_d^{(n)}/K)} / U_{\chi(I_d^{(n)}/K)} - Q_{d,n}^{(n)}/K^{nr}_{(n,d)+1}),\]

for some \((n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}\).

**4.3. Main theorems.** We can now state and prove the main theorem, that is, the ramification theorem for the non-abelian local reciprocity map \(\Phi_{K}^{(\varphi)}\). In order to do so, we first prove the ramification theorem for the generalized arrow \(\Phi_{K}^{(\varphi)} : G_{K} \to \nabla_{K}^{0}\).

**Theorem 4.15 (Ramification theorem for \(\Phi_{K}^{(\varphi)}\).** For any increasing net \(w = (w_{(n,d)})\) in \(\mathbb{R}_{\geq -1}\) satisfying \(0 \leq \psi_{F_d}^{(\varphi)} / K^{nr}_{(n,d)}(w_{(n,d)}) \in \mathbb{Z}\) for every \((n, d)\)
in \( \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \), we have the inclusion

\[
\phi_{c, K}^{(\sigma)}(G_w^{u} - G_{w'}^{u'}) \subseteq 1 \nabla_{K}^{o, w} - 1 \nabla_{K}^{o, w'}.
\]

Proof. Let \( w \) be as in the assumptions. Let

\[
\sigma = (\sigma_{d, n})_{d, n} \in \lim_{\rightarrow (n, d)} G(d, n)^{w(n, d)} = G_w^{u}.
\]

Clearly, by Corollary 4.12(i), \( \phi_{c, K}^{(\sigma)}(\sigma) \in 1 \nabla_{K}^{o, w} \). By Lemma 4.7 the condition \( \sigma \in G_w^{u} - G_{w'}^{u'} \) is equivalent to the existence of a pair \( (n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \) satisfying \( \sigma_{d, n} \in G(d, n)_{\psi_{d}^{(\sigma)}}(w(n, d)) - G(d, n)_{\psi_{d}^{(\sigma)}}(w(n, d) + 1) \). Therefore, by the ramification theorem for the generalized arrow \( \phi_{c, K}^{(\sigma)}(\sigma) \), stated in (4.1),

\[
\phi_{c, K}^{(\sigma)}(\sigma_{d, n}) \in \{1_{K} \times N_{K_{d}}/K K_{d}^{n} \times \}
\times \left( (U_{K}^{(\sigma)}(I_{d}^{(n)}/K))_{n}^{(\sigma)}(w(n, d)) U_{X}(I_{d}^{(n)}/K) / U_{X}(I_{d}^{(n)}/K) \right)
- \psi_{d}^{(\sigma)}(w(n, d) + 1) U_{X}(I_{d}^{(n)}/K) / U_{X}(I_{d}^{(n)}/K),
\]

which proves, by Lemma 4.14(i), that

\[
\phi_{c, K}^{(\sigma)}(\sigma) \in 1 \nabla_{K}^{o, w} - 1 \nabla_{K}^{o, w'} \nabla_{K}^{o, w'}.
\]

Theorem 4.16 (Ramification theorem for \( \Phi_{c, K}^{(\sigma)} \)). For any increasing net \( w = (w(n, d)) \) in \( \mathbb{R}_{\geq 1} \) satisfying \( 0 \leq \psi_{d}^{(\sigma)}(w(n, d)) \subseteq \mathbb{Z} \) for every \( (n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \), we have the inclusion

\[
\Phi_{c, K}^{(\sigma)}(G_w^{u} - G_{w'}^{u'}) \subseteq 1 \nabla_{K}^{u} - Q_{K}^{u'}.
\]

Proof. Let \( w \) be as above. Let \( \sigma = (\sigma_{d, n})_{d, n} \in \lim_{\rightarrow (n, d)} G(d, n)^{w(n, d)} = G_w^{u} \).

Clearly, by Corollary 4.12(ii), \( \Phi_{c, K}^{(\sigma)}(\sigma) \in 1 \nabla_{K}^{u} \). By Lemma 4.7 the condition \( \sigma \in G_w^{u} - G_{w'}^{u'} \) is equivalent to the existence of a pair \( (n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \) satisfying \( \sigma_{d, n} \in G(d, n)_{\psi_{d}^{(\sigma)}}(w(n, d)) - G(d, n)_{\psi_{d}^{(\sigma)}}(w(n, d) + 1) \). Therefore, by the ramification theorem for the generalized Fesenko reciprocity map \( \Phi_{c, K}^{(\sigma)}(\sigma) \), stated in (4.2),

\[
\Phi_{c, K}^{(\sigma)}(\sigma_{d, n}) \in \{1_{K} \times N_{K_{d}^{n}}/K K_{d}^{n} \times \}
\times \left( (U_{K}^{(\sigma)}(I_{d}^{(n)}/K))_{n}^{(\sigma)}(w(n, d)) Y_{d}^{(n)}/K_{d}^{n} / Y_{d}^{(n)}/K_{d}^{n} - Q_{d}^{(n)}/K_{d}^{n}(w(n, d) + 1),
\]
which proves, by Lemma 4.14(ii), that
\[
\Phi^{(\phi)}_K(\sigma) \in \mathcal{N}_K^{w} - Q_K^{w'}.
\]

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References


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