ACTA ARITHMETICA
* (201*)

Ramification theory in non-abelian local class field theory

by

KÂZIM ILHAN IKEDA and EROL SERBEST (Istanbul)

To the memory of I. M. Gelfand

1. Introduction. Let K be a local field, that is, a complete discrete valuation field with finite residue class field κ_K of $q = p^f$ elements. For technical reasons, throughout the paper we shall assume that the multiplicative group $\boldsymbol{\mu}_p(K^{\mathrm{sep}})$ of all pth roots of unity in K^{sep} satisfies $\boldsymbol{\mu}_p(K^{\mathrm{sep}}) \subset K$. Fix a Lubin–Tate splitting φ over K. That is, we fix an extension φ of the Frobenius automorphism of K^{nr} to K^{sep} (for details, cf. [Ko-dS]). In a sequence of papers [Ik-Se-1, Ik-Se-2, Ik-Se-3], following the idea of Fesenko developed in [Fes-1, Fes-2, Fes-3], we have constructed the *non-abelian local reciprocity map* $\boldsymbol{\Phi}_K^{(\varphi)}$ *for* K, which is an isomorphism from the absolute Galois group G_K of K onto a certain topological group $\nabla_K^{(\varphi)}$ which depends on the choice of the Lubin–Tate splitting φ.

The aim of the present paper is to study the ramification-theoretic properties of the map $\boldsymbol{\Phi}_K^{(\varphi)}$. We prove (in Theorems 4.15 and 4.16) that $\boldsymbol{\Phi}_K^{(\varphi)}$ is *compatible* with the refined higher ramification "filtration" of the absolute Galois group G_K of K (cf. 4.1) and the refined "filtration" of $\nabla_K^{(\varphi)}$ (cf. 4.2).

The organization of the paper is as follows. In Section 2, we collect the necessary results from the theory of local fields. In Section 3, we briefly review the main results of [Ik-Se-2] on the generalized Fesenko reciprocity map, and then sketch the construction of the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}$ following [Ik-Se-3]. In the last section, we first introduce the refined filtrations on G_K and on $\nabla_K^{(\varphi)}$ and then prove the main results of the paper, which are stated as Theorems 4.15 and 4.16.

2010 *Mathematics Subject Classification*: Primary 11S37.
Key words and phrases: local fields, higher-ramification theory, APF-extensions, Fontaine–Wintenberger fields of norms, generalized Fesenko reciprocity map, non-abelian local reciprocity map, non-abelian local class field theory.

2. Preliminaries on local fields. In this section, we shall briefly review the necessary background material from the theory of local fields.

2.1. Local fields. Throughout this work, K will denote a local field, that is, a complete discrete valuation field with finite residue class field $O_K/\mathfrak{p}_K =: \kappa_K$ of $q_K = q = p^f$ elements with p a prime number, where O_K denotes the ring of integers in K with the unique maximal ideal $\mathfrak{p}_K$. Let $\boldsymbol{\nu}_K$ denote the corresponding normalized valuation on K (normalized by $\boldsymbol{\nu}_K(K^\times) = \mathbb{Z}$). As usual, the unit group of K is denoted by U_K and the ith higher unit group of K by U_K^i, where $0 \leq i \in \mathbb{Z}$.

Let K^{sep} denote a fixed separable closure of K, and K^{nr} the maximal unramified extension of K inside K^{sep}. The unique extension of $\boldsymbol{\nu}_K$ to K^{sep} will be denoted by $\widetilde{\boldsymbol{\nu}}$, and for any sub-extension L/K of K^{sep}/K, the normalized form of the valuation $\widetilde{\boldsymbol{\nu}}|_L$ on L will be denoted by $\boldsymbol{\nu}_L$. The completion of K^{nr} with respect to the valuation $\boldsymbol{\nu}_{K^{\mathrm{nr}}}$ will be denoted by $\widetilde{K}$. For any separable extension L/K, we put $\widetilde{L} := L\widetilde{K}$.

Let G_K denote the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. The topological generator of $\mathrm{Gal}(K^{\mathrm{nr}}/K)$, which is the Frobenius automorphism of K, is denoted by $\varphi_K = \varphi$ (if there is no risk of confusion). Any extension of the automorphism $\varphi : K^{\mathrm{nr}} \to K^{\mathrm{nr}}$ to K^{sep} is called a *Lubin–Tate splitting over* K and is again denoted by φ.

We further assume that the multiplicative group $\boldsymbol{\mu}_p(K^{\mathrm{sep}})$ of pth roots of unity in K^{sep} satisfies

$$\boldsymbol{\mu}_p(K^{\mathrm{sep}}) \subset K. \tag{2.1}$$

2.2. Local Artin reciprocity map. Let G_K^{ab} denote the maximal abelian Hausdorff quotient group G_K/G_K' of the topological group G_K, where G_K' denotes the closure of the first commutator subgroup $[G_K, G_K]$ of G_K.

Recall that abelian local class field theory for the local field K establishes a unique natural algebraic and topological isomorphism

$$\mathsf{Art}_K : G_K^{\mathrm{ab}} \xrightarrow{\sim} \widehat{K^\times},$$

called the *local Artin reciprocity map* of K, where the topological group $\widehat{K^\times}$ denotes the pro-finite completion of the multiplicative group $K^\times$, satisfying certain properties. In particular, for an abelian extension L/K, and for every integer $0 \leq i \in \mathbb{Z}$ and real number $\nu \in (i-1, i]$,

$$x \in U_K^i \mathcal{N}_L \;\Leftrightarrow\; \mathsf{Art}_{L/K}^{-1}(x) \in \mathrm{Gal}(L/K)^\nu,$$

where $x \in \widehat{K^\times}$. Here, $\mathcal{N}_L$ denotes the closed subgroup of $\widehat{K^\times}$ defined to be the intersection $\mathcal{N}_L = \bigcap_E N_{E/K}\widehat{E^\times}$, where E runs over all finite extensions of K inside L.

In what follows, we shall briefly review the higher ramification subgroups in the upper numbering of the absolute Galois group G_K of K.

2.3. A brief review of ramification theory. The main reference that we follow closely here is [Ik-Se-1].

For a finite separable extension L/K, and for any $\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}})$, introduce

$$i_{L/K}(\sigma) := \min_{x \in O_L} \{\boldsymbol{\nu}_L(\sigma(x) - x)\},$$

put

$$\gamma_t := \#\{\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}}) : i_{L/K}(\sigma) \geq t + 1\},$$

for $-1 \leq t \in \mathbb{R}$, and define the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$, the *Hasse–Herbrand transition function* of the extension L/K, by

$$\varphi_{L/K}(u) := \begin{cases} \int\limits_0^u \dfrac{\gamma_t}{\gamma_0}\, dt, & 0 \leq u \in \mathbb{R}, \\ u, & -1 \leq u \leq 0. \end{cases}$$

It is well-known that $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a continuous, increasing, piecewise linear function, and it establishes a homeomorphism $\mathbb{R}_{\geq -1} \xrightarrow{\approx} \mathbb{R}_{\geq -1}$. Let $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be its inverse.

Assume that L is a finite Galois extension over K with Galois group $\mathrm{Gal}(L/K) =: G$. The normal subgroup G_u of G defined by

$$G_u = \{\sigma \in G : i_{L/K}(\sigma) \geq u + 1\}$$

for $-1 \leq u \in \mathbb{R}$ is called the uth *ramification group of G in the lower numbering*, and has order γ_u. Note the inclusion $G_{u'} \subseteq G_u$ for every pair $-1 \leq u, u' \in \mathbb{R}$ satisfying $u \leq u'$. The family $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ induces a filtration on G, called the *lower ramification filtration* of G. A *break in this filtration* is defined to be any number $u \in \mathbb{R}_{\geq -1}$ satisfying $G_u \neq G_{u+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$. The function $\psi_{L/K} = \varphi_{L/K}^{-1} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ induces the *upper ramification filtration* $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ on G by setting

$$G^v := G_{\psi_{L/K}(v)},$$

or equivalently, by setting

$$G^{\varphi_{L/K}(u)} = G_u$$

for $-1 \leq v, u \in \mathbb{R}$; here G^v is called the vth *upper ramification group* of G. A *break* in the upper filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G is defined to be any number $v \in \mathbb{R}_{\geq -1}$ satisfying $G^v \neq G^{v+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$.

Remark 2.1. We list the basic properties of lower and upper ramification filtrations on G. In what follows, F/K denotes a sub-extension of L/K and H denotes the Galois group $\mathrm{Gal}(L/F)$.

(i) The lower numbering on G passes to the subgroup H of G in the sense that

$$H_u = H \cap G_u \quad \text{for } -1 \leq u \in \mathbb{R}.$$

(ii) If $H \lhd G$, then the upper numbering on G passes to the quotient G/H:

$$(G/H)^v = G^v H/H \quad \text{for } -1 \leq v \in \mathbb{R}.$$

(iii) The Hasse–Herbrand function and its inverse satisfy the transitive law

$$\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F} \quad \text{and} \quad \psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}.$$

If L/K is an *infinite* Galois extension with Galois group $\mathrm{Gal}(L/K) = G$, which is a topological group under the respective Krull topology, define the upper ramification filtration $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ on G by the projective limit

$$G^v := \varprojlim_{K\subseteq F\subset L} \mathrm{Gal}(F/K)^v \tag{2.2}$$

over the transition morphisms $t_F^{F'}(v) : \mathrm{Gal}(F'/K)^v \to \mathrm{Gal}(F/K)^v$, which are essentially the restriction morphisms from F' to F, defined naturally by the diagram

$$\begin{tikzcd}
\mathrm{Gal}(F/K)^v & & \mathrm{Gal}(F'/K)^v \arrow[ll, "t_F^{F'}(v)"'] \arrow[dl, "\text{can.}"] \\
& \mathrm{Gal}(F'/K)^v\mathrm{Gal}(F'/F)/\mathrm{Gal}(F'/F) \arrow[ul, "\substack{\text{isomorphism}\\ \text{introduced in (ii)}}"] &
\end{tikzcd} \tag{2.3}$$

induced from (ii), as $K \subseteq F \subseteq F' \subseteq L$ runs over all finite Galois extensions F and F' over K inside L. The topological subgroup G^v of G is called the vth *ramification group of G in the upper numbering.* Note the inclusion $G^{v'} \subseteq G^v$ for every pair $-1 \leq v, v' \in \mathbb{R}$ satisfying $v \leq v'$ via the commutativity of the square

$$\begin{tikzcd}
\mathrm{Gal}(F/K)^v & \mathrm{Gal}(F'/K)^v \arrow[l, "t_F^{F'}(v)"'] \\
\mathrm{Gal}(F/K)^{v'} \arrow[u, "\text{inc.}"] & \mathrm{Gal}(F'/K)^{v'} \arrow[l, "t_F^{F'}(v')"'] \arrow[u, "\text{inc.}"']
\end{tikzcd} \tag{2.4}$$

for every chain $K \subseteq F \subseteq F' \subseteq L$ of finite Galois extensions F and F' over K inside L. Observe that:

(iv) $G^{-1} = G$ and G^0 is the inertia subgroup of G.

(v) $\bigcap_{v\in\mathbb{R}_{\geq -1}} G^v = \langle 1_G \rangle$.

(vi) G^v is a closed subgroup of G, with respect to the Krull topology, for $-1 \le v \in \mathbb{R}$.

In this setting, a number $-1 \le v \in \mathbb{R}$ is said to be a *break in the upper ramification filtration* $\{G^v\}_{v\in\mathbb{R}_{\ge -1}}$ of G, if v is a break in the upper filtration of some finite quotient G/H for some $H \triangleleft G$. Let $\mathcal{B}_{L/K}$ denote the set of all numbers $v \in \mathbb{R}_{\ge -1}$ which occur as breaks in the upper ramification filtration of G. Then:

(vii) (Hasse–Arf theorem) $\mathcal{B}_{K^{\mathrm{ab}}/K} \subseteq \mathbb{Z} \cap \mathbb{R}_{\ge -1}$.
(viii) $\mathcal{B}_{K^{\mathrm{sep}}/K} \subseteq \mathbb{Q} \cap \mathbb{R}_{\ge -1}$.

2.4. APF-extensions. As in the previous section, let $\{G_K^v\}_{v\in\mathbb{R}_{\ge -1}}$ denote the upper ramification filtration of the absolute Galois group G_K of K, and let R^v denote the fixed field $(K^{\mathrm{sep}})^{G_K^v}$ of the vth upper ramification subgroup G_K^v of G_K in K^{sep} for $-1 \le v \in \mathbb{R}$.

Definition 2.2. An extension L/K is called an *APF-extension* (APF is the shortening for "arithmétiquement profinie") if one of the following equivalent conditions is satisfied:

(i) $G_K^v G_L$ is open in G_K for every $-1 \le v \in \mathbb{R}$,
(ii) $(G_K : G_K^v G_L) < \infty$ for every $-1 \le v \in \mathbb{R}$,
(iii) $L \cap R^v$ is a finite extension over K for every $-1 \le v \in \mathbb{R}$.

Note that if L/K is an APF-extension, then $[\kappa_L : \kappa_K] < \infty$.

Now, let L/K be an APF-extension. Set $G_L^0 = G_L \cap G_K^0$, and define

$$\psi_{L/K}(v) = \begin{cases} \int_0^v (G_K^0 : G_L^0 G_K^x)\, dx, & 0 \le v \in \mathbb{R}, \\ v, & -1 \le v \le 0. \end{cases} \tag{2.5}$$

Then the map $v \mapsto \psi_{L/K}(v)$ for $v \in \mathbb{R}_{\ge -1}$, which is well-defined for the APF-extension L/K, defines a continuous, strictly increasing and piecewise linear bijection $\psi_{L/K} : \mathbb{R}_{\ge -1} \to \mathbb{R}_{\ge -1}$.

We denote the inverse of $\psi_{L/K}$ by $\varphi_{L/K}$. Thus, if L/K is a (not necessarily finite) Galois APF-extension, then we can define the higher ramification subgroups in the lower numbering $\mathrm{Gal}(L/K)_u$ of $\mathrm{Gal}(L/K)$, for $-1 \le u \in \mathbb{R}$, by setting

$$\mathrm{Gal}(L/K)_u := \mathrm{Gal}(L/K)^{\varphi_{L/K}(u)}.$$

Remark 2.3. Note that:

(i) In case L/K is a finite separable extension, which is clearly an APF-extension by Definition 2.2, the function $\psi_{L/K} : \mathbb{R}_{\ge -1} \to \mathbb{R}_{\ge -1}$ coincides with the inverse of the Hasse–Herbrand transition function of L/K introduced in the previous section.

(ii) If L/K is a finite separable extension and L'/L is an APF-extension, then L'/K is an APF-extension, and the transitivity rules for the functions $\psi_{L'/K}, \varphi_{L'/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ hold:

$$\psi_{L'/K} = \psi_{L'/L} \circ \psi_{L/K}, \qquad \varphi_{L'/K} = \varphi_{L/K} \circ \varphi_{L'/L}.$$

3. Non-abelian local reciprocity map. In this section, we shall review the theory developed in [Ik-Se-2, Ik-Se-3]. Fix a Lubin–Tate splitting φ over K.

3.1. Generalized Fesenko reciprocity map. For an infinite APF-Galois extension L/K with residue class degree $[\kappa_L : \kappa_K] = d$ and with $K \subset L \subset K_{\varphi^d}$, denote the field of norms corresponding to L/K by $\mathbb{X}(L/K)$ and the completion of the maximal unramified extension $\mathbb{X}(L/K)^{\mathrm{nr}}$ of $\mathbb{X}(L/K)$ by $\widetilde{\mathbb{X}}(L/K)$ (for details, [Fe-Vo], [Fo-Wi-1, Fo-Wi-2] and [Win]), and set $L_0 = L \cap K^{\mathrm{nr}}$. There exists a bijective 1-cocycle

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{3.1}$$

called the *generalized Fesenko reciprocity map* for the extension L/K, defined by the composition

$$\begin{tikzcd} \mathrm{Gal}(L/K) \arrow[r, "\boldsymbol{\phi}^{(\varphi)}_{L/K}"] \arrow[dr, dotted, "\boldsymbol{\Phi}^{(\varphi)}_{L/K}"'] & K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \arrow[d, "(\mathrm{id}_{K^\times/N_{L_0/K}L_0^\times}, c_{L/L_0})"] \\ & K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \end{tikzcd} \tag{3.2}$$

Here,

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \tag{3.3}$$

is an injective 1-cocycle called, following [Ik-Se-2], the *generalized arrow defined for the extension L/K*, and defined by

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma) = (\pi_K^m N_{L_0/K}L_0^\times, \phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)), \tag{3.4}$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $0 \leq m \in \mathbb{Z}$ is the integer satisfying $\sigma|_{L_0} = \varphi^m|_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$, and for any $\tau \in \mathrm{Gal}(L/L_0)$, the value $\phi^{(\varphi^d)}_{L/L_0}(\tau)$ of the arrow defined for the extension L/L_0 at τ is defined by [Fes-1, Fes-2, Fes-3] and [Ik-Se-1]. Namely, $\phi^{(\varphi^d)}_{L/L_0}(\tau) = U_\tau.U_{\mathbb{X}(L/L_0)}$ provided that $U_\tau \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, which is unique modulo $U_{\mathbb{X}(L/L_0)}$, solves the equation $U^{1-\varphi^d} = \Pi^{\tau-1}_{\varphi^d;L/L_0}$, where $\Pi_{\varphi^d;L/L_0}$ is the canonical prime element of the local field $\mathbb{X}(L/L_0)$ defined in Lemmas 1.2 and 1.3 of [Ik-Se-2]. For the definition of the group $U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$ and its subgroups $U_{\mathbb{X}(L/L_0)}$ and Y_{L/L_0} satisfying the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$ we refer the reader to [Fes-1, Fes-2, Fes-3] and [Ik-Se-1]. In the commutative triangle (3.2), the arrow

$$c_{L/L_0} : U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \tag{3.5}$$

is the canonical map defined by the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$. Recall that (cf. [Fes-1, Fes-2, Fes-3] and [Ik-Se-1]) the composition $c_{L/L_0} \circ \phi^{(\varphi^d)}_{L/L_0} = \Phi^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is the Fesenko reciprocity map for the extension L/L_0. Thus, for $\sigma \in \mathrm{Gal}(L/K)$, the value $\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma)$ is defined by

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma) = (\pi_K^m N_{L_0/K} L_0^{\times}, \Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)), \tag{3.6}$$

where $0 \leq m \in \mathbb{Z}$ satisfies $\sigma|_{L_0} = \varphi^m|_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$.

Define a composition law $*$ on $\mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{L/K})$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}).(\overline{b}, \overline{V})^{(\boldsymbol{\phi}^{(\varphi)}_{L/K})^{-1}((\overline{a}, \overline{U}))} \tag{3.7}$$

for every $\overline{a} = a.N_{L_0/K} L_0^{\times}$, $\overline{b} = b.N_{L_0/K} L_0^{\times} \in K^{\times}/N_{L_0/K} L_0^{\times}$ with $a, b \in K^{\times}$ and $\overline{U} = U.U_{\mathbb{X}(L/K)}$, $\overline{V} = V.U_{\mathbb{X}(L/K)} \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ with $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $\mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{L/K})$ is defined by $(\overline{b}, \overline{V})^{\sigma} = (\overline{b}, \overline{V}^{\varphi^{-m}\sigma})$. Then $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ is a topological group under $*$, and $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{L/K}), \tag{3.8}$$

where the topological group structure on $\mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{L/K})$ is defined with respect to the binary operation $*$ defined by (3.7). Likewise, define a composition law, again denoted by $*$, on $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}).(\overline{b}, \overline{V})^{(\boldsymbol{\Phi}^{(\varphi)}_{L/K})^{-1}((\overline{a}, \overline{U}))} \tag{3.9}$$

for every $\overline{a} = a.N_{L_0/K} L_0^{\times}$, $\overline{b} = b.N_{L_0/K} L_0^{\times} \in K^{\times}/N_{L_0/K} L_0^{\times}$ with $a, b \in K^{\times}$ and $\overline{U} = U.Y_{L/L_0}$, $\overline{V} = V.Y_{L/L_0} \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ with $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined by $(\overline{b}, \overline{V})^{\sigma} = (\overline{b}, \overline{V}^{\varphi^{-m}\sigma})$. Then $K^{\times}/N_{L_0/K} L_0^{\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is a topological group under $*$, and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{3.10}$$

where the topological group structure on $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined with respect to the binary operation $*$ defined by (3.9).

The mappings $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ have the following basic properties.

(i) For an infinite Galois sub-extension M/K of L/K such that $[\kappa_M : \kappa_K] = d'$ and $K \subset M \subset K_{\varphi^{d'}}$ for some $d' \mid d$, the square

$$\begin{CD} \mathrm{Gal}(L/K) @>{\boldsymbol{\phi}^{(\varphi)}_{L/K}}>> K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \\ @V{\mathrm{res}_M}VV @VV{(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M})}V \\ \mathrm{Gal}(M/K) @>{\boldsymbol{\phi}^{(\varphi)}_{M/K}}>> K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)} \end{CD} \tag{3.11}$$

is commutative, where the right vertical arrow is defined by

$$(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}) : (\overline{a}, \overline{U}) \mapsto (e^{\mathrm{CFT}}_{L_0/M_0}(\overline{a}), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})) \tag{3.12}$$

for every $(\overline{a}, \overline{U}) \in K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$. Here,

$$\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}$$

is the Coleman norm map from L to M defined by equations (2.22) and (2.23) of [Ik-Se-2]. Likewise, the square

$$\begin{CD} \mathrm{Gal}(L/K) @>{\boldsymbol{\Phi}^{(\varphi)}_{L/K}}>> K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \\ @V{\mathrm{res}_M}VV @VV{(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M})}V \\ \mathrm{Gal}(M/K) @>{\boldsymbol{\Phi}^{(\varphi)}_{M/K}}>> K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/M_0} \end{CD} \tag{3.13}$$

is commutative, where the right vertical arrow is defined by

$$(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}) : (\overline{a}, \overline{U}) \mapsto (e^{\mathrm{CFT}}_{L_0/M_0}(\overline{a}), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})) \tag{3.14}$$

for $(\overline{a}, \overline{U}) \in K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$. Here, $\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/M_0}$ is the Coleman norm map from L to M defined by Lemma 2.21 together with equations (2.47) and (2.48) of [Ik-Se-2]. Moreover, the arrow $e^{\mathrm{CFT}}_{L_0/M_0} : K^\times/N_{L_0/K}L_0^\times \to K^\times/N_{M_0/K}M_0^\times$ appearing in both commutative diagrams is the natural inclusion defined via the existence theorem of local class field theory.

(ii) For each $0 \leq i \in \mathbb{R}$, introduce the subgroups $(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^i$ of the field $\widetilde{\mathbb{X}}(L/K)$ by $(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^i = U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \cap U^i_{\widetilde{\mathbb{X}}(L/K)}$. For each $0 \leq n \in \mathbb{Z}$, as in equation (5.42) of [Ik-Se-1], let

$$Q^n_{L/L_0} = c_{L/L_0}((U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \cap \mathrm{im}(\phi^{(\varphi^d)}_{L/L_0})), \tag{3.15}$$

which is a subgroup of $(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^n Y_{L/L_0}/Y_{L/L_0}$. Here, the canonical homomorphism c_{L/L_0} introduced in (3.5) is defined by equation (5.35) of [Ik-Se-1]. Now, the ramification theorem for the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ yields, for $0 \leq n \in \mathbb{Z}$, the inclusion

$$\begin{aligned} \boldsymbol{\phi}^{(\varphi)}_{L/K}&(\mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n)} - \mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n+1)}) \\ &\subseteq \langle 1_{K^\times/N_{L_0/K}L_0^\times}\rangle \times ((U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \\ &\qquad - (U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^{n+1} U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}), \end{aligned} \tag{3.16}$$

and the ramification theorem for the generalized Fesenko reciprocity map $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ gives, for $0 \leq n \in \mathbb{Z}$, the inclusion

$$\begin{aligned} \boldsymbol{\Phi}^{(\varphi)}_{L/K}&(\mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n)} - \mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n+1)}) \\ &\subseteq \langle 1_{K^\times/N_{L_0/K}L_0^\times}\rangle \times ((U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^n Y_{L/L_0}/Y_{L/L_0} - Q^{n+1}_{L/L_0}), \end{aligned} \tag{3.17}$$

where, for $0 \leq u \in \mathbb{R}$, $\mathrm{Gal}(L/K)_u$ denotes the uth ramification subgroup in the lower numbering of the Galois group $\mathrm{Gal}(L/K)$ corresponding to the infinite APF-Galois extension L/K.

Remark 3.1. In fact, ramification theorems for $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ stated in (3.16) and (3.17) can be simplified as follows. For $0 \leq n \in \mathbb{Z}$, as $\varphi_{L/K}(n) = \varphi_{L_0/K} \circ \varphi_{L/L_0}(n)$ and $L_0 = L \cap K^{\mathrm{nr}}$, it follows that $\varphi_{L/K}(n) = \varphi_{L/L_0}(n)$. Therefore, (3.16) can be reformulated as

$$\begin{aligned} \boldsymbol{\phi}^{(\varphi)}_{L/K}&(\mathrm{Gal}(L/K)_n - \mathrm{Gal}(L/K)_{n+1}) \subseteq \langle 1_{K^\times/N_{L_0/K}L_0^\times}\rangle \\ &\times ((U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} - (U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^{n+1} U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}), \end{aligned} \tag{3.18}$$

and (3.17) can be reformulated as

$$\begin{aligned} \boldsymbol{\Phi}^{(\varphi)}_{L/K}&(\mathrm{Gal}(L/K)_n - \mathrm{Gal}(L/K)_{n+1}) \\ &\subseteq \langle 1_{K^\times/N_{L_0/K}L_0^\times}\rangle \times ((U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^n Y_{L/L_0}/Y_{L/L_0} - Q^{n+1}_{L/L_0}). \end{aligned} \tag{3.19}$$

Finally, the following remark is in order.

Remark 3.2. We do not need assumption (2.1) on the local field K to define the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ by (3.4). For details, cf. [Ik-Se-2].

3.2. Construction of the non-abelian local reciprocity map. For each $1 \leq d \in \mathbb{Z}$, let K_{φ^d} denote the fixed field of $\varphi^d \in G_K$. Observe that $K^{\mathrm{sep}} = K^{\mathrm{nr}} K_{\varphi^d}$ and $K_d^{\mathrm{nr}} = K^{\mathrm{nr}} \cap K_{\varphi^d}$, where K_d^{nr} denotes the unique unramified extension over K of degree d. Now, for each $1 \leq n, d \in \mathbb{Z}$, let $\Gamma_d^{(n)} := \Gamma_d^{(n)}(K, \varphi)$ be a Galois extension over K, which is the unique maximal n-abelian extension ([1]) of K_d^{nr} in K_{φ^d}. Note that

$$\bigcup_{1 \leq d \in \mathbb{Z}} \Gamma_d^{(n)} = (K^{\mathrm{nr}})^{n\text{-ab}}, \tag{3.20}$$

where $(K^{\mathrm{nr}})^{n\text{-ab}}$ denotes the "n-abelian closure" of K^{nr} in K^{sep}. Thus, it also follows that

$$\bigcup_{1 \leq n \in \mathbb{Z}} \bigcup_{1 \leq d \in \mathbb{Z}} \Gamma_d^{(n)} = K^{\mathrm{sep}}. \tag{3.21}$$

Moreover, for each pair (n, d) of positive integers, $\Gamma_d^{(n)}$ is an APF-extension over K.

Now, the absolute Galois group G_K of the local field K is the projective limit

$$G_K = \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$$

over the restriction morphisms

$$r_{(n',d')}^{(n,d)} : \mathrm{Gal}(\Gamma_d^{(n)}/K) \to \mathrm{Gal}(\Gamma_{d'}^{(n')}/K)$$

for $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $n' \leq n$ and $d' \mid d$ (which is equivalent to $\Gamma_{d'}^{(n')} \subseteq \Gamma_d^{(n)}$). Note that, for each $1 \leq n, d \in \mathbb{Z}$, the APF-Galois extension $\Gamma_d^{(n)}$ over K has the residue class degree d. Therefore, the generalized Fesenko theory developed in [Ik-Se-2] can be applied to the extensions of the form $\Gamma_d^{(n)}/K$, which would enable us to construct the generalized arrow $\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}$ and the generalized Fesenko reciprocity map $\boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}$, for every pair $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. Then using property (i) for the collections $\{\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}\}_{(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}}$ and $\{\boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}\}_{(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}}$, and passing to the projective limits, we get the *generalized arrow* $\boldsymbol{\phi}_K^{(\varphi)}$ *for the local field* K and the *non-abelian local reciprocity map* $\boldsymbol{\Phi}_K^{(\varphi)}$ *for the local field* K respectively.

To be more precise, we first introduce the following notation to simplify the discussion. In what follows, L/K denotes an infinite APF-Galois extension such that $[\kappa_L : \kappa_K] = d$ and $K \subset L \subset K_{\varphi^d}$.

([1]) Recall that by an *n-abelian extension* over a field F, we mean a Galois extension E/F whose Galois group $\mathrm{Gal}(E/F)$ has a trivial nth commutator subgroup $\mathrm{Gal}(E/F)^{(n)}$.

NOTATION 3.3. For an infinite Galois sub-extension M/K of L/K such that $[\kappa_M : \kappa_K] = d'$ and $K \subset M \subset K_{\varphi^{d'}}$ provided that $d' \mid d$, we let

(i) $\mathcal{C}^o_{L/M}$ denote the map $(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M})$ defined by (3.11) and (3.12),

(ii) $\mathcal{C}_{L/M}$ denote the map $(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M})$ defined by (3.13) and (3.14).

Recall that

$$\mathcal{C}^o_{L/M} : K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to K^\times/N_{M_0/K}M_0^\times \times U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}$$

and

$$\mathcal{C}_{L/M} : K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to K^\times/N_{M_0/K}M_0^\times \times U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/M_0}$$

are homomorphisms of the underlying abelian groups. Moreover, for the valued fields L and M as above, let F/K be an infinite Galois sub-extension of M/K satisfying $K \subset F \subset K_{\varphi^{d''}}$ with $[\kappa_F : \kappa_K] = d''$ where $d'' \mid d'$. If we set $F_0 = F \cap K^{\mathrm{nr}}$, the following equalities hold:

(i) $\mathcal{C}^o_{L/M} = \mathrm{id}$ and $\mathcal{C}_{L/M} = \mathrm{id}$, if $L = M$.

(ii) $\mathcal{C}^o_{L/F} = \mathcal{C}^o_{M/F} \circ \mathcal{C}^o_{L/M}$ and $\mathcal{C}_{L/F} = \mathcal{C}_{M/F} \circ \mathcal{C}_{L/M}$.

It follows that the systems

$$\{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times} \times U^\diamond_{\widetilde{\mathbb{X}}(\varGamma_d^{(n)}/K)}/U_{\mathbb{X}(\varGamma_d^{(n)}/K)}; \mathcal{C}^o_{\varGamma_d^{(n)}/\varGamma_{d'}^{(n')}}\}_{\substack{n' \leq n \\ d' \mid d}} \tag{3.22}$$

and

$$\{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times} \times U^\diamond_{\widetilde{\mathbb{X}}(\varGamma_d^{(n)}/K)}/Y_{\varGamma_d^{(n)}/K_d^{\mathrm{nr}}}; \mathcal{C}_{\varGamma_d^{(n)}/\varGamma_{d'}^{(n')}}\}_{\substack{n' \leq n \\ d' \mid d}} \tag{3.23}$$

are projective. Let

$$\begin{aligned}\nabla_K^{(\varphi),o} = \nabla_K^o &= \varprojlim_{(n,d)} K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times} \times U^\diamond_{\widetilde{\mathbb{X}}(\varGamma_d^{(n)}/K)}/U_{\mathbb{X}(\varGamma_d^{(n)}/K)} \\ &= \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} U^\diamond_{\widetilde{\mathbb{X}}(\varGamma_d^{(n)}/K)}/U_{\mathbb{X}(\varGamma_d^{(n)}/K)}\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\nabla_K^{(\varphi)} = \nabla_K &= \varprojlim_{(n,d)} K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times} \times U^\diamond_{\widetilde{\mathbb{X}}(\varGamma_d^{(n)}/K)}/Y_{\varGamma_d^{(n)}/K_d^{\mathrm{nr}}} \\ &= \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} U^\diamond_{\widetilde{\mathbb{X}}(\varGamma_d^{(n)}/K)}/Y_{\varGamma_d^{(n)}/K_d^{\mathrm{nr}}}\end{aligned} \tag{3.25}$$

be the projective limits of the systems (3.22) and (3.23) respectively. The limits $\nabla_K^{(\varphi),o}$ and $\nabla_K^{(\varphi)}$, or ∇_K^o and ∇_K respectively if there is no risk of confusion, depend on the choice of a Lubin–Tate splitting φ over K.

Note that ∇_K^o and ∇_K have natural topological G_K-module structures, where the G_K-action on ∇_K^o and on ∇_K is defined by

$$((\overline{a}_{d,n},\overline{U}_{d,n}))_{d,n}^{\sigma} = ((\overline{a}_{d,n},\overline{U}_{d,n})^{\sigma|_{\Gamma_d^{(n)}}})_{d,n} \tag{3.26}$$

for every coherent sequence $((\overline{a}_{d,n},\overline{U}_{d,n}))_{d,n}$ from ∇_K^o or from ∇_K, and for every $\sigma \in G_K = \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$.

For any two pairs (n,d) and (n',d') satisfying $n' \leq n$ and $d' \mid d$, the square

$$\begin{tikzcd}
U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)} \arrow[r, "c_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}"] \arrow[d, "\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}"'] & U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}/Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}} \arrow[d, "\widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}"] \\
U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_{d'}^{(n')}/K)}/U_{\mathbb{X}(\Gamma_{d'}^{(n')}/K)} \arrow[r, "c_{\Gamma_{d'}^{(n')}/K_{d'}^{\mathrm{nr}}}"] & U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_{d'}^{(n')}/K)}/Y_{\Gamma_{d'}^{(n')}/K_{d'}^{\mathrm{nr}}}
\end{tikzcd} \tag{3.27}$$

is commutative. Therefore, the topological G_K-modules ∇_K^o and ∇_K are related to each other by a topological G_K-module homomorphism

$$c_K := \varprojlim_{(n,d)} (\mathrm{id}_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}, c_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}) : \nabla_K^o \to \nabla_K \tag{3.28}$$

defined by the commutativity of the diagram (3.27).

Therefore, there exists an injective map

$$\boldsymbol{\phi}_K^{(\varphi)} = \varprojlim_{(n,d)} \boldsymbol{\phi}^{(\varphi)}_{\Gamma_d^{(n)}/K} : G_K \to \nabla_K^o \tag{3.29}$$

defined by

$$\boldsymbol{\phi}_K^{(\varphi)}((\sigma_{d,n})_{d,n}) = (\boldsymbol{\phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}(\sigma_{d,n}))_{d,n} \tag{3.30}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K) = G_K$, and a bijective map

$$\boldsymbol{\Phi}_K^{(\varphi)} = \varprojlim_{(n,d)} \boldsymbol{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K} : G_K \to \nabla_K \tag{3.31}$$

defined by

$$\boldsymbol{\Phi}_K^{(\varphi)}((\sigma_{d,n})_{d,n}) = (\boldsymbol{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}(\sigma_{d,n}))_{d,n} \tag{3.32}$$

for every coherent sequence $(\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K) = G_K$. Moreover, the injective mapping $\boldsymbol{\phi}_K^{(\varphi)} : G_K \to \nabla_K^o$ is a 1-cocycle, that is, for $\sigma, \tau \in G_K$ with respective coherent sequences $(\sigma_{d,n})_{d,n}, (\tau_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$,

$$\boldsymbol{\phi}_K^{(\varphi)}(\sigma\tau) = \boldsymbol{\phi}_K^{(\varphi)}(\sigma)\boldsymbol{\phi}_K^{(\varphi)}(\tau)^{\sigma}. \tag{3.33}$$

Also the bijective mapping $\boldsymbol{\Phi}_K^{(\varphi)}: G_K \to \nabla_K$ is a 1-cocycle, i.e., for $\sigma, \tau \in G_K$ with respective coherent sequences $(\sigma_{d,n})_{d,n}, (\tau_{d,n})_{d,n} \in \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$,

$$\boldsymbol{\Phi}_K^{(\varphi)}(\sigma\tau) = \boldsymbol{\Phi}_K^{(\varphi)}(\sigma)\boldsymbol{\Phi}_K^{(\varphi)}(\tau)^\sigma. \tag{3.34}$$

DEFINITION 3.4. The injective 1-cocycle $\boldsymbol{\phi}_K^{(\varphi)}: G_K \to \nabla_K^o$ is called the *generalized arrow for* K, and the bijective 1-cocyle $\boldsymbol{\Phi}_K^{(\varphi)}: G_K \to \nabla_K$ is called the *non-abelian local reciprocity map of* K.

The 1-cocycles $\boldsymbol{\phi}_K^{(\varphi)}$ and $\boldsymbol{\Phi}_K^{(\varphi)}$ are related to each other by

$$\boldsymbol{\Phi}_K^{(\varphi)} = c_K \circ \boldsymbol{\phi}_K^{(\varphi)}. \tag{3.35}$$

4. Ramification theory. Now, by Theorems 2.7 and 2.20 of [Ik-Se-2] (cf. also Remark 3.1 in Section 3), the ramification theorems for the generalized arrow $\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}$ and for the generalized Fesenko reciprocity map $\boldsymbol{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}$ give, for $0 \le w \in \mathbb{Z}$, the inclusions

$$\begin{aligned} &\boldsymbol{\phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}(\mathrm{Gal}(\Gamma_d^{(n)}/K)_w - \mathrm{Gal}(\Gamma_d^{(n)}/K)_{w+1}) \\ &\quad\subseteq \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}\rangle \times ((U^\diamond_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^w U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)} \\ &\qquad - (U^\diamond_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{w+1} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)}), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\boldsymbol{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}(\mathrm{Gal}(\Gamma_d^{(n)}/K)_w - \mathrm{Gal}(\Gamma_d^{(n)}/K)_{w+1}) \\ &\quad\subseteq \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}\rangle \times ((U^\diamond_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^w Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}/Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}} - Q^{w+1}_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}), \end{aligned} \tag{4.2}$$

where $\mathrm{Gal}(\Gamma_d^{(n)}/K)_w$ denotes the wth higher ramification subgroup in the lower numbering of the Galois group $\mathrm{Gal}(\Gamma_d^{(n)}/K)$ corresponding to the infinite APF-Galois extension $\Gamma_d^{(n)}/K$.

The aim of this section is to state and prove ramification theorems for the generalized arrow $\boldsymbol{\phi}_K^{(\varphi)}: G_K \to \nabla_K^o$ and for the non-abelian local reciprocity map $\boldsymbol{\Phi}_K^{(\varphi)}: G_K \to \nabla_K$.

4.1. Higher ramification subgroups of G_K in the upper numbering. To simplify the discussion, we introduce the following notation.

NOTATION 4.1. For every $1 \le d, n \in \mathbb{Z}$, the Galois group $\mathrm{Gal}(\Gamma_d^{(n)}/K)$ is denoted by $G(d,n)$. Moreover, for any $-1 \le w \in \mathbb{R}$, $G(d,n)^w$ denotes the wth ramification subgroup of $G(d,n)$ in the upper numbering.

Proposition 4.2. *For $(n',d'),(n,d)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$ satisfying $n'\leq n$ and $d'\,|\,d$, and for $0\leq w\in\mathbb{Z}$,*

$$\psi_{\Gamma_{d'}^{(n')}/K_{d'}^{\mathrm{nr}}}(w)\leq\psi_{\Gamma_{d}^{(n)}/K_{d}^{\mathrm{nr}}}(w). \tag{4.3}$$

Proof. As $\Gamma_{d'}^{(n')}$ and $\Gamma_{d}^{(n)}$ are APF-extensions over the field K, for every $-1<x\in\mathbb{R}$, setting $G^0_{\Gamma_{d'}^{(n')}}=G^0_K\cap G_{\Gamma_{d'}^{(n')}}$ and $G^0_{\Gamma_{d}^{(n)}}=G^0_K\cap G_{\Gamma_{d}^{(n)}}$, we have $(G^0_K:G^x_KG^0_{\Gamma_{d'}^{(n')}})<\infty$ and $(G^0_K:G^x_KG^0_{\Gamma_{d}^{(n)}})<\infty$ (cf. [Fo-Wi-1, Fo-Wi-2, Win]). Now, if $n'\leq n$ and $d'\,|\,d$, then $\Gamma_{d'}^{(n')}\subseteq\Gamma_{d}^{(n)}$. Therefore,

$$(G^0_K:G^x_KG^0_{\Gamma_{d'}^{(n')}})\leq(G^0_K:G^x_KG^0_{\Gamma_{d}^{(n)}})\leq\infty,$$

as $G_{\Gamma_{d}^{(n)}}\subseteq G_{\Gamma_{d'}^{(n')}}$. Hence, for $0\leq w\in\mathbb{Z}$,

$$\psi_{\Gamma_{d'}^{(n')}/K}(w)=\int_0^w(G^0_K:G^x_KG^0_{\Gamma_{d'}^{(n')}})\,dx\leq\int_0^w(G^0_K:G^x_KG^0_{\Gamma_{d}^{(n)}})\,dx=\psi_{\Gamma_{d}^{(n)}/K}(w).$$

Now, the desired inequality follows, because

$$\psi_{\Gamma_{d'}^{(n')}/K}(w)=\psi_{\Gamma_{d'}^{(n')}/K_{d'}^{\mathrm{nr}}}\circ\psi_{K_{d'}^{\mathrm{nr}}/K}(w)=\psi_{\Gamma_{d'}^{(n')}/K_{d'}^{\mathrm{nr}}}(w)$$

and likewise

$$\psi_{\Gamma_{d}^{(n)}/K}(w)=\psi_{\Gamma_{d}^{(n)}/K_{d}^{\mathrm{nr}}}\circ\psi_{K_{d}^{\mathrm{nr}}/K}(w)=\psi_{\Gamma_{d}^{(n)}/K_{d}^{\mathrm{nr}}}(w).\ \blacksquare$$

Remark 4.3. Note that Proposition 4.2 is more generally true in the following setting. Let L be an infinite APF-Galois extension over K satisfying $K\subset L\subset K_{\varphi^d}$ with $[\kappa_L:\kappa_K]=d$, and M/K be an infinite Galois sub-extension of L/K satisfying $K\subset M\subset K_{\varphi^{d'}}$ with $[\kappa_M:\kappa_K]=d'$, where $d'\,|\,d$. Then, for $0\leq w\in\mathbb{Z}$,

$$\psi_{M/M_0}(w)\leq\psi_{L/L_0}(w),$$

where $L_0=L\cap K^{\mathrm{nr}}$ and $M_0=M\cap K^{\mathrm{nr}}$. The proof follows the same lines.

It is well-known that, for a *fixed* $-1\leq w\in\mathbb{R}$, the projective limit

$$G^w_K:=\varprojlim_{(n,d)}G(d,n)^w \tag{4.4}$$

over the restriction morphisms

$$r^{(n,d)}_{(n',d')}:G(d,n)^w\to G(d',n')^w$$

for $(n,d),(n',d')\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$ satisfying $n'\leq n$ and $d'\,|\,d$ defines a subgroup G^w_K of the absolute Galois group G_K, and we have the following definition.

Definition 4.4. For $-1\leq w\in\mathbb{R}$, the group G^w_K is called the wth *higher ramification subgroup of G_K in the upper numbering.*

However, it turns out that we need a *finer* upper ramification "filtration" of G_K. Let $\underline{w} := (w_{(n,d)})$ be a *net* in $\mathbb{R}_{\geq -1}$ always assumed to be indexed over the directed set $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, where $(n', d') \preceq (n, d)$ if $n' \leq n$ and $d' \,|\, d$ for $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. Furthermore, assume that the net $\underline{w}$ in $\mathbb{R}_{\geq -1}$ is *increasing*, that is, $w_{(n',d')} \leq w_{(n,d)}$ if $(n', d') \preceq (n, d)$. In case $\underline{w} = (w_{(n,d)})$ in $\mathbb{R}_{\geq -1}$ is *constant*, that is, $w_{(n,d)} = c$ for every $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, the net $\underline{w}$ will be simply denoted by c.

Note that, for an increasing net $\underline{w}$ in $\mathbb{R}_{\geq -1}$, the projective limit

$$G_K^{\underline{w}} := \varprojlim_{(n,d)} G(d, n)^{w_{(n,d)}} \tag{4.5}$$

over the restriction morphisms

$$r^{(n,d)}_{(n',d')} : G(d, n)^{w_{(n,d)}} \to G(d', n')^{w_{(n,d)}} \hookrightarrow G(d', n')^{w_{(n',d')}}$$

for $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $n' \leq n$ and $d' \,|\, d$ defines a subgroup $G_K^{\underline{w}}$ of the absolute Galois group G_K, and we have the following definition.

Definition 4.5. For an increasing net $\underline{w}$ in $\mathbb{R}_{\geq -1}$, the group $G_K^{\underline{w}}$ is called the $\underline{w}$th *higher ramification subgroup of G_K in the upper numbering.*

Definition 4.6. Let $\underline{w} = (w_{(n,d)})$ be an increasing net in $\mathbb{R}_{\geq -1}$. The net $\underline{w}'$ in $\mathbb{R}_{\geq -1}$ defined by

$$w'_{(n,d)} = \varphi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)}) + 1), \tag{4.6}$$

for every pair (n, d), which is clearly an increasing net in $\mathbb{R}_{\geq -1}$, is called the *successor* of $\underline{w}$.

Note that, for any increasing net $\underline{w}$ in $\mathbb{R}_{\geq -1}$, we have the inclusion

$$G_K^{\underline{w}'} \subseteq G_K^{\underline{w}}, \tag{4.7}$$

because $G(d, n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1} \subseteq G(d, n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}$ for every pair (n, d). The proof of the following lemma is clear.

Lemma 4.7. *For any increasing net $\underline{w}$ in $\mathbb{R}_{\geq -1}$ and for $\sigma = (\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} G(d, n)^{w_{(n,d)}} = G_K^{\underline{w}}$, the following two conditions are equivalent.*

(i) $\sigma \in G_K^{\underline{w}} - G_K^{\underline{w}'}$.
(ii) $\sigma_{d,n} \in G(d, n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} - G(d, n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1}$ *for* $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

4.2. The groups $\nabla_K^{o,\underline{w}}$ and $\nabla_K^{\underline{w}}$ for an increasing net $\underline{w}$ in $\mathbb{R}_{\geq -1}$. The following proposition is central to what follows.

Proposition 4.8. *Let L be an infinite APF-Galois extension over K satisfying $K \subset L \subset K_{\varphi^d}$ with $[\kappa_L : \kappa_K] = d$, and M/K be an infinite Galois* *sub-extension of L/K satisfying $K \subset M \subset K_{\varphi^{d'}}$ with $[\kappa_M : \kappa_K] = d'$, where $d' \mid d$. Then*

$$\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}((U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^w) \subseteq (U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)})^w \tag{4.8}$$

for every $0 \leq w \in \mathbb{Z}$.

Proof. For $0 \leq w \in \mathbb{Z}$, let $\alpha = (\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in (U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^w$. That is, the norm coherent sequence $\alpha = (\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ satisfies

$$\nu_{\widetilde{\mathbb{X}}(L/K)}((\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} - 1_{\widetilde{\mathbb{X}}(L/K)}) = \nu_{\widetilde{K}}(\alpha_{\widetilde{K}} - 1) \geq w,$$

where the equality follows from the definition of addition on $\widetilde{\mathbb{X}}(L/K)$ and the valuation $v_{\widetilde{\mathbb{X}}(L/K)}$ on $\widetilde{\mathbb{X}}(L/K)$. Thus, by the definition of the mapping $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M} : \mathbb{X}(L/K)^\times \to \mathbb{X}(M/K)^\times$, it follows that

$$\begin{aligned}\nu_{\widetilde{\mathbb{X}}(M/K)}&(\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}((\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}}) - 1_{\widetilde{\mathbb{X}}(M/K)}) \\ &= \nu_{\widetilde{K}}(\alpha_{\widetilde{K}}^{(1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)})^2} - 1) = \nu_{\widetilde{K}}(\alpha_{\widetilde{K}}^{f(L/M)^2} - 1),\end{aligned}$$

as $\alpha = (\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ and the $\widetilde{K}$-coordinate of α satisfies $\alpha_{\widetilde{K}} \in U_{L_0}$, where $L_0 = L \cap K^{\mathrm{nr}}$. Thus,

$$\begin{aligned}\nu_{\widetilde{\mathbb{X}}(M/K)}&(\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}((\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}}) - 1_{\widetilde{\mathbb{X}}(M/K)}) \\ &= \nu_{\widetilde{K}}(\alpha_{\widetilde{K}} - 1) + \nu_{\widetilde{K}}\Big(\sum_{0 \leq \ell \lneq f(L/M)^2} \alpha_{\widetilde{K}}^{\ell}\Big) \geq w,\end{aligned}$$

which shows that $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}((\alpha_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}}) \in U^{w}_{\widetilde{\mathbb{X}}(M/K)}$. Combining this with the property (ii) of equation (2.21) in [Ik-Se-2] yields the assertion. ■

Notation 4.9. Let L be an infinite APF-Galois extension over K satisfying $K \subset L \subset K_{\varphi^d}$ with $[\kappa_L : \kappa_K] = d$. For $0 \leq w \in \mathbb{R}$, let

$$(\nabla^{(\varphi),o}_{L/K})^w = \nabla^{o,w}_{L/K} = K^\times / N_{L_0/K} L_0^\times \times (U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^w U_{\mathbb{X}(L/K)} / U_{\mathbb{X}(L/K)}, \tag{4.9}$$

$$(\nabla^{(\varphi)}_{L/K})^w = \nabla^{w}_{L/K} = K^\times / N_{L_0/K} L_0^\times \times (U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)})^w Y_{L/L_0} / Y_{L/L_0}. \tag{4.10}$$

Therefore, by Remark 4.3, for the local fields L and M as in Proposition 4.8, and for $0 \leq w_{L/K}, w_{M/K} \in \mathbb{R}$ satisfying $w_{M/K} \leq w_{L/K}$, the map $\mathcal{C}^o_{L/M}$ introduced in Notation 3.3(i) restricts to

$$\mathcal{C}^o_{L/M} : \nabla^{o,\psi_{L/L_0}(w_{L/K})}_{L/K} \to \nabla^{o,\psi_{M/M_0}(w_{M/K})}_{M/K},$$

and the map $\mathcal{C}_{L/M}$ introduced in Notation 3.3(ii) restricts to

$$\mathcal{C}_{L/M} : \nabla^{\psi_{L/L_0}(w_{L/K})}_{L/K} \to \nabla^{\psi_{M/M_0}(w_{M/K})}_{M/K}.$$

Thus, the following corollary follows directly.

COROLLARY 4.10. *For an increasing net $\underline{w} = (w_{(n,d)})$ in $\mathbb{R}_{\geq 0}$, the systems*

$$\{\nabla^{o,\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}_{\Gamma_d^{(n)}/K}; \mathcal{C}^o_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}\}_{\substack{n'\leq n\\ d'|d}} \tag{4.11}$$

and

$$\{\nabla^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}_{\Gamma_d^{(n)}/K}; \mathcal{C}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}\}_{\substack{n'\leq n\\ d'|d}} \tag{4.12}$$

are projective.

Proof. Follows from the projectivity of the systems (3.22) and (3.23), and from Proposition 4.8 combined with Proposition 4.2. ∎

For any increasing net $\underline{w}$ in $\mathbb{R}_{\geq 0}$, let

$$\begin{aligned}(\nabla_K^{(\varphi),o})^{\underline{w}} := \nabla_K^{o,\underline{w}} &= \varprojlim_{(n,d)} \nabla^{o,\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}_{\Gamma_d^{(n)}/K} \\ &= \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}(\nabla_K^{(\varphi)})^{\underline{w}} := \nabla_K^{\underline{w}} &= \varprojlim_{(n,d)} \nabla^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}_{\Gamma_d^{(n)}/K} \\ &= \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}/Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}\end{aligned} \tag{4.14}$$

be the projective limits of the systems (4.11) and (4.12) respectively. These limits $(\nabla_K^{(\varphi),o})^{\underline{w}}$ and $(\nabla_K^{(\varphi)})^{\underline{w}}$, or $\nabla_K^{o,\underline{w}}$ and $\nabla_K^{\underline{w}}$ if there is no risk of confusion, depend on the choice of a Lubin–Tate splitting φ over K.

LEMMA 4.11. *For (n,d), (n',d') satisfying $n' \leq n$, $d' \mid d$, and for $0 \leq w_{(n,d)}, w_{(n',d')} \in \mathbb{R}$ satisfying $w_{(n',d')} \leq w_{(n,d)}$ and $0 \leq \psi_{\Gamma_d^{(n)}/K}(w_{(n,d)})$, $\psi_{\Gamma_{d'}^{(n')}/K}(w_{(n',d')}) \in \mathbb{Z}$, the squares*

$$\begin{CD} G(d,n)^{w_{(n,d)}} @>{\boldsymbol{\phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}}>> A^{(U)}_{(n,d)} \\ @V{r^{(n,d)}_{(n',d')}}VV @VV{\mathcal{C}^o_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}}V \\ G(d',n')^{w_{(n',d')}} @>{\boldsymbol{\phi}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}}>> A^{(U)}_{(n',d')} \end{CD} \tag{4.15}$$

and

$$\begin{tikzcd} G(d,n)^{w_{(n,d)}} \arrow[r, "\boldsymbol{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}"] \arrow[d, "r^{(n,d)}_{(n',d')}"'] & A^{(Y)}_{(n,d)} \arrow[d, "\mathcal{C}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}"] \\ G(d,n)^{w_{(n',d')}} \arrow[r, "\boldsymbol{\Phi}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}"] & A^{(Y)}_{(n',d')} \end{tikzcd} \tag{4.16}$$

are commutative, where for $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ *and* $0 \leq w_{(n,d)} \in \mathbb{R}$ *satisfying* $0 \leq \psi_{\Gamma_d^{(n)}/K}(w_{(n,d)}) \in \mathbb{Z}$,

$$A^{(U)}_{(n,d)} := \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}} \rangle \times (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K}(w_{(n,d)})} U_{\mathbb{X}(\Gamma_d^{(n)}/K)} / U_{\mathbb{X}(\Gamma_d^{(n)}/K)},$$

$$A^{(Y)}_{(n,d)} := \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}} \rangle \times (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}} / Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}.$$

Proof. Follows from Proposition 4.2, and the basic property (iii) of ramification theory of $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ together with the basic property (i) of $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ and $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ stated in Section 3. ∎

For any increasing net $\underline{w} = (w_{(n,d)})$ in $\mathbb{R}_{\geq 0}$, let ${}_1\nabla^{o,\underline{w}}_K$ denote the kernel of the projection $\mathrm{Pr}_1 : \nabla^{o,\underline{w}}_K \to \widehat{\mathbb{Z}}$, and ${}_1\nabla^{\underline{w}}_K$ denote the kernel of the projection $\mathrm{Pr}_1 : \nabla^{\underline{w}}_K \to \widehat{\mathbb{Z}}$. An immediate consequence of Lemma 4.11 is

Corollary 4.12. *Let* $\underline{w}$ *be any increasing net in* $\mathbb{R}_{\geq 0}$ *and* $\sigma \in G^{\underline{w}}_K$. *Then:*

(i) $\boldsymbol{\phi}^{(\varphi)}_K(\sigma) \in {}_1\nabla^{o,\underline{w}}_K$.
(ii) $\boldsymbol{\Phi}^{(\varphi)}_K(\sigma) \in {}_1\nabla^{\underline{w}}_K$.

Now, for an increasing net $\underline{w} = (w_{(n,d)})$ in $\mathbb{R}_{\geq 0}$, introduce

$$Q^{\underline{w}}_K = c_K({}_1\nabla^{o,\underline{w}}_K \cap \mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_K)), \tag{4.17}$$

where $c_K : \nabla^o_K \to \nabla_K$ is the canonical map defined by (3.28). Note that $c_K({}_1\nabla^{o,w}_K) = {}_1\nabla^w_K$ by the commutativity of the square (3.27) and by Propositions 4.2 and 4.8.

Lemma 4.13. *For an increasing net* $\underline{w} = (w_{(n,d)})$ *in* $\mathbb{R}_{\geq 0}$,

$$\left\{ Q^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}} ; \mathcal{N}^{\mathrm{Coleman}}_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}} \right\}_{\substack{n' \leq n \\ d' \mid d}} \tag{4.18}$$

is a projective system and its projective limit is

$$Q^{\underline{w}}_K = \langle 1_{\widehat{\mathbb{Z}}} \rangle \times \varprojlim_{(n,d)} Q^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})}_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}. \tag{4.19}$$

Proof. The projectivity of (4.18) follows from the projectivity of the system $\{\mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}), \mathcal{C}^o_{\Gamma_d^{(n)}/\Gamma_{d'}^{(n')}}\}_{\substack{n'\leq n\\ d'|d}}$ combined with Proposition 4.8 and (3.27). Moreover, the equality (4.19) follows from (3.28) and (3.27). ∎

In the lemma below, whose proof is clear, $\underline{w}'$ denotes the successor of $\underline{w}$.

LEMMA 4.14. (i) *For any increasing net* $\underline{w} = (w_{(n,d)})$ *in* $\mathbb{R}_{\geq -1}$ *and for an element* $u = (u_{d,n})_{d,n} = ((1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}, \overline{U}_{d,n}))_{d,n}$ *of*

$$ {}_1\nabla_K^{o,\underline{w}} = \langle 1_{\widehat{\mathbb{Z}}}\rangle \times \varprojlim_{(n,d)} (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)}, $$

we have: $u \in {}_1\nabla_K^{o,\underline{w}} - {}_1\nabla_K^{o,\underline{w}'}$ *if and only if*

$$ \begin{aligned} u_{d,n} = (1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}, \overline{U}_{d,n}) \in \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}\rangle \\ \times ((U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)} \\ - (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)}) \end{aligned} $$

for some $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

(ii) *For any increasing net* $\underline{w} = (w_{(n,d)})$ *in* $\mathbb{R}_{\geq -1}$ *and for an element* $u = (u_{(d,n)})_{(d,n)} = ((1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}, \overline{U}_{d,n}))_{d,n}$ *of*

$$ {}_1\nabla_K^{\underline{w}} = \langle 1_{\widehat{\mathbb{Z}}}\rangle \times \varprojlim_{(n,d)} (U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}/Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}, $$

we have: $u \in {}_1\nabla_K^{\underline{w}} - Q_K^{\underline{w}'}$ *if and only if*

$$ \begin{aligned} u_{(d,n)} = (1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}, \overline{U}_{d,n}) \in \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}\rangle \\ \times ((U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}/Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}} - Q_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1}) \end{aligned} $$

for some $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

4.3. Main theorems. We can now state and prove the main theorem, that is, the *ramification theorem for the non-abelian local reciprocity map* $\boldsymbol{\Phi}_K^{(\varphi)}$. In order to do so, we first prove the *ramification theorem for the generalized arrow* $\boldsymbol{\phi}_K^{(\varphi)} : G_K \to \nabla_K^o$.

THEOREM 4.15 (Ramification theorem for $\boldsymbol{\phi}_K^{(\varphi)}$). *For any increasing net* $\underline{w} = (w_{(n,d)})$ *in* $\mathbb{R}_{\geq -1}$ *satisfying* $0 \leq \psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)}) \in \mathbb{Z}$ *for every* (n,d)

in $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, *we have the inclusion*

$$\boldsymbol{\phi}_K^{(\varphi)}(G_K^{\underline{w}} - G_K^{\underline{w}'}) \subseteq {}_1\nabla_K^{o,\underline{w}} - {}_1\nabla_K^{o,\underline{w}'}. \tag{4.20}$$

Proof. Let $\underline{w}$ be as in the assumptions. Let

$$\sigma = (\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} G(d,n)^{w_{(n,d)}} = G_K^{\underline{w}}.$$

Clearly, by Corollary 4.12(i), $\boldsymbol{\phi}_K^{(\varphi)}(\sigma) \in {}_1\nabla_K^{o,\underline{w}}$. By Lemma 4.7, the condition $\sigma \in G_K^{\underline{w}} - G_K^{\underline{w}'}$ is equivalent to the existence of a pair $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $\sigma_{d,n} \in G(d,n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} - G(d,n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1}$. Therefore, by the ramification theorem for the generalized arrow $\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}$, stated in (4.1),

$$\begin{aligned}\boldsymbol{\phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}(\sigma_{d,n}) \in \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}\rangle \\ \times ((U^\diamond_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)} \\ - (U^\diamond_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1} U_{\mathbb{X}(\Gamma_d^{(n)}/K)}/U_{\mathbb{X}(\Gamma_d^{(n)}/K)}),\end{aligned}$$

which proves, by Lemma 4.14(i), that

$$\boldsymbol{\phi}_K^{(\varphi)}(\sigma) \in {}_1\nabla_K^{o,\underline{w}} - {}_1\nabla_K^{o,\underline{w}'}. \ \blacksquare$$

THEOREM 4.16 (Ramification theorem for $\boldsymbol{\Phi}_K^{(\varphi)}$). *For any increasing net* $\underline{w} = (w_{(n,d)})$ *in* $\mathbb{R}_{\geq -1}$ *satisfying* $0 \leq \psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)}) \in \mathbb{Z}$ *for every* $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, *we have the inclusion*

$$\boldsymbol{\Phi}_K^{(\varphi)}(G_K^{\underline{w}} - G_K^{\underline{w}'}) \subseteq {}_1\nabla_K^{\underline{w}} - Q_K^{\underline{w}'}. \tag{4.21}$$

Proof. Let $\underline{w}$ be as above. Let $\sigma = (\sigma_{d,n})_{d,n} \in \varprojlim_{(n,d)} G(d,n)^{w_{(n,d)}} = G_K^{\underline{w}}$. Clearly, by Corollary 4.12(ii), $\boldsymbol{\Phi}_K^{(\varphi)}(\sigma) \in {}_1\nabla_K^{\underline{w}}$. By Lemma 4.7, the condition $\sigma \in G_K^{\underline{w}} - G_K^{\underline{w}'}$ is equivalent to the existence of a pair $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $\sigma_{d,n} \in G(d,n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} - G(d,n)_{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1}$. Therefore, by the ramification theorem for the generalized Fesenko reciprocity map $\boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}$, stated in (4.2),

$$\begin{aligned}\boldsymbol{\Phi}_{\Gamma_d^{(n)}/K}^{(\varphi)}(\sigma_{d,n}) \in \langle 1_{K^\times/N_{K_d^{\mathrm{nr}}/K}K_d^{\mathrm{nr}\times}}\rangle \\ \times ((U^\diamond_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)})^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}/Y_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}} - Q_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}^{\psi_{\Gamma_d^{(n)}/K_d^{\mathrm{nr}}}(w_{(n,d)})+1}),\end{aligned}$$

which proves, by Lemma 4.14(ii), that

$$\boldsymbol{\Phi}_K^{(\varphi)}(\sigma) \in {}_1\nabla_K^{w} - Q_K^{w'}. \ \blacksquare$$

Acknowledgements. The first named author (K. I. I.) is partly supported by the TÜBİTAK grant 107T728.

References

[Fes-1] I. B. Fesenko, *Local reciprocity cycles*, in: Invitation to Higher Local Fields, I. B. Fesenko and M. Kurihara (eds.), Geom. Topol. Monogr. 3, Coventry, 2000, 293–298.

[Fes-2] —, *Nonabelian local reciprocity maps*, in: Class Field Theory—Its Centenary and Prospect (Tokyo, 1998), K. Miyake (ed.), Adv. Stud. Pure Math. 30, Math. Soc. Japan, 2001, 63–78.

[Fes-3] —, *On the image of noncommutative local reciprocity map*, Homology Homotopy Appl. 7 (2005), 53–62.

[Fe-Vo] I. B. Fesenko and S. V. Vostokov, *Local Fields and Their Extensions*, 2nd ed., Transl. Math. Monogr. 121, Amer. Math. Soc., Providence, RI, 2002.

[Fo-Wi-1] J.-M. Fontaine et J.-P. Wintenberger, *Le "corps des normes" de certaines extensions algébriques de corps locaux*, C. R. Acad. Sci. Paris Sér. A Math. 288 (1979), 367–370.

[Fo-Wi-2] —, —, *Extensions algébriques et corps des normes des extensions APF des corps locaux*, ibid. 288 (1979), 441–444.

[Ik-Se-1] K. I. Ikeda and E. Serbest, *Fesenko reciprocity map*, Algebra i Analiz 20 (2008), no. 3, 112–162 (in Russian).

[Ik-Se-2] —, —, *Generalized Fesenko reciprocity map*, ibid. 20 (2008), no. 4, 118–159 (in Russian).

[Ik-Se-3] —, —, *Non-abelian local reciprocity law*, Manuscripta Math., to appear.

[Ko-dS] H. Koch and E. de Shalit, *Metabelian local class field theory*, J. Reine Angew. Math. 478 (1996), 85–106.

[Win] J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. 46 (1983), 59–89.

Kâzim Ilhan Ikeda, Erol Serbest
Department of Mathematics
Yeditepe University
26 Ağustos Yerleşimi, İnönü Mah.
Kayışdağı Cad., 34755 Kadıköy
Istanbul, Turkey
E-mail: ilhan.ikeda@yeditepe.edu.tr
erol.serbest@yeditepe.edu.tr

Received on 20.11.2009
and in revised form on 24.3.2010 (6221)