ON THE RELATIONSHIP BETWEEN THE GENERALIZED FESENKO AND THE LAUBIE RECIPROCITY MAPS

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–To the memory of our teacher Cemal Koç–

Abstract. In this paper, we study the relationship between the non-abelian local class field theories in the sense of Fesenko and in the sense of Laubie. For a local field K, we construct a topological isomorphism $\mathscr{C}_K^{(\varphi)}$ between the groups $\nabla_K^{(\varphi)}$ and $\mathscr{G}_K^{(\varphi)}$, which are the generalizations of the topological group $\widehat{K^\times}$ of abelian local class field theory of K to the setting of non-abelian local class field theories of K in the sense of Fesenko and in the sense of Laubie respectively. We show that, the isomorphism $\mathscr{C}_K^{(\varphi)}$ transfers these non-abelian local class field theories to each other, is functorial with respect to the base change, and compatible with the "refined filtrations" of $\nabla_K^{(\varphi)}$ and of $\mathscr{G}_K^{(\varphi)}$.

Contents

1. Introduction and the statement of main results

Let K be a local field with finite residue-class field $\kappa_K := O_K/\mathfrak{p}_K$ of $q = p^f$ elements, where O_K denotes the ring of integers in K with the unique prime ideal $\mathfrak{p}_K$ (for details about local fields, we refer the reader to [4]). Fix a *Lubin-Tate splitting* $\varphi_K = \varphi$ *over* K. Namely, we fix a K-automorphism φ of K^{sep} such that $\varphi\mid_{K^{nr}}$ is the Frobenius automorphism Frob_K of K. The Lubin-Tate splitting φ over K uniquely determines a *Lubin-Tate labelling* $\mathscr{L}_\varphi$ *over* K. That is, φ uniquely

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determines a *"norm-coherent"* collection $\mathscr{L}_\varphi = \{\pi_L\}_L$ consisting of prime elements π_L of $\widetilde{L} = \widehat{L^{nr}}$ (= the completion of L^{nr}), for every finite extension L over K (look at [11] for the definition and basic properties on Lubin-Tate labellings over K).

The non-abelian local class field theory for K (developed in a series of papers [7, 8, 9, 10]), which extends the idea of Fesenko (introduced in [1, 2, 3]), establishes an algebraic and topological isomorphism

$$\mathbf{\Phi}_K^{(\varphi)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi)}$$

between the absolute Galois group G_K of K and a certain topological group $\nabla_K^{(\varphi)}$ which depends on K and on the choice of the Lubin-Tate splitting φ over K. The construction of the topological group $\nabla_K^{(\varphi)}$, which will be defined explicitly in Section 3, involves the theory of APF-extensions of K and the fields of norms construction of Fontaine and Wintenberger (look at [5, 6, 13] for details). Moreover, the isomorphism $\mathbf{\Phi}_K^{(\varphi)}$, which is called the *non-abelian local reciprocity law of* K, is "natural" in the sense that properties such as "existence", "functoriality" and a certain "ramification theoretic" property are all satisfied.

Another, albeit related approach to the problem of constructing the non-abelian local class field theory, carried out by Laubie in [12], generalizes the work of Koch and de Shalit in [11] and establishes an algebraic and topological isomorphism

$$\mathbf{\Lambda}_K^{(\varphi)} : G_K \xrightarrow{\sim} \mathscr{G}_K^{(\varphi)}$$

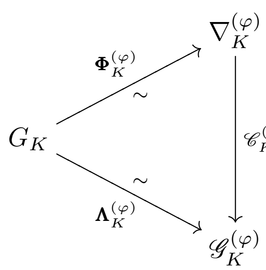
between the absolute Galois group G_K of K and a certain topological group $\mathscr{G}_K^{(\varphi)}$ which depends on K and again on the choice of the Lubin-Tate splitting φ over K. The topological group $\mathscr{G}_K^{(\varphi)}$ is defined in terms of APF-extensions of K and the fields of norms theory combined with the Koch-de Shalit construction (look at [11, 12, 13] for details). Following [12], we shall explicitly define $\mathscr{G}_K^{(\varphi)}$ in Section 4. Moreover, the isomorphism $\mathbf{\Lambda}_K^{(\varphi)}$, which we have decided to call the *Laubie reciprocity law of* K, is "natural" in the sense that properties such as "existence", "functoriality" and a certain "ramification theoretic" property are all satisfied.

The aim of this work, which generalizes Remark 5 of [2] and Section 3.7 of [12], and which is a complement to [9, 10, 12], is to investigate the relationship between the arrows $\mathbf{\Phi}_K^{(\varphi)}$ and and $\mathbf{\Lambda}_K^{(\varphi)}$ defined for K. More precisely, we shall construct an *explicitly defined* (that is, defined independently from $\mathbf{\Phi}_K^{(\varphi)}$ and $\mathbf{\Lambda}_K^{(\varphi)}$) topological isomorphism

$$\mathscr{C}_K^{(\varphi)} : \nabla_K^{(\varphi)} \xrightarrow{\sim} \mathscr{G}_K^{(\varphi)}, \tag{1.1}$$

called the *"comparison isomorphism" with respect to* K, which is essentially the composite map $\mathbf{\Lambda}_K^{(\varphi)} \circ \left(\mathbf{\Phi}_K^{(\varphi)}\right)^{-1}$, so that the triangle

(1.2)



becomes commutative. Moreover, we shall prove that the isomorphism (1.1) is functorial with respect to the base change and respects the "refined filtrations" on $\nabla_K^{(\varphi)}$ and on $\mathscr{G}_K^{(\varphi)}$ (look at [10] and Section 8). That is, we shall recover the Laubie reciprocity law $\mathbf{\Lambda}_K^{(\varphi)}$ of the local field K from the non-abelian local reciprocity law $\mathbf{\Phi}_K^{(\varphi)}$ of K introduced in [9] together with the "comparison isomorphism" $\mathscr{C}_K^{(\varphi)}$ with respect to K that we shall introduce in this work (or recover $\mathbf{\Phi}_K^{(\varphi)}$ from $\mathbf{\Lambda}_K^{(\varphi)}$ and $\left(\mathscr{C}_K^{(\varphi)}\right)^{-1}$). So, one can use the full power of both theories [9, 10] and [12] together via the arrow $\mathscr{C}_K^{(\varphi)} : \nabla_K^{(\varphi)} \to \mathscr{G}_K^{(\varphi)}$ defined with respect to K, as it unifies Laubie's theory over K and the theory developed in [9, 10] over K.

It turns out that, in order to construct the topological isomorphism (1.1) satisfying (1.2) with respect to K with the above mentioned properties, it suffices to consider a certain type of extension $\Gamma_d^{(n)}/K$ for each $1 \leq n, d \in \mathbb{Z}$. We shall briefly recall the definition and the basic properties of the collection of extensions $\left\{\Gamma_d^{(n)}/K\right\}_{n,d}$ where $1 \leq n, d \in \mathbb{Z}$ in Section 2. Next, in Sections 3 and 4, we shall briefly review the constructions of the *generalized Fesenko* and *Laubie reciprocity laws*

$$\begin{array}{ccc} & & \nabla_{L/K}^{(\varphi)} \\ & \overset{\mathbf{\Phi}_{L/K}^{(\varphi)}}{\nearrow \sim} & \\ \mathrm{Gal}(L/K) & & \\ & \underset{\mathbf{\Lambda}_{L/K}^{(\varphi)}}{\searrow \sim} & \\ & & \mathscr{G}_{L/K}^{(\varphi)} \end{array}$$

for an infinite APF-Galois extension L/K with residue degree $[\kappa_L : \kappa_K] = d$ satisfying $K \subset L \subset K_{\varphi^d}$, namely for an infinite φ^d-compatible APF-Galois extension L/K in the sense of [11, 12], respectively.

In Section 5, we shall prove the following theorem.

Theorem 1.1. *Let L/K be an infinite φ^d-compatible APF-Galois extension. Then, there exists an explicitly defined topological isomorphism*

$$\mathscr{C}_{L/K}^{(\varphi)} : \nabla_{L/K}^{(\varphi)} \xrightarrow{\sim} \mathscr{G}_{L/K}^{(\varphi)}, \tag{1.3}$$

called the "comparison isomorphism" with respect to the extension L/K, satisfying

$$\mathscr{C}_{L/K}^{(\varphi)} = \mathbf{\Lambda}_{L/K}^{(\varphi)} \circ \left(\mathbf{\Phi}_{L/K}^{(\varphi)}\right)^{-1}. \tag{1.4}$$

Moreover, specializing to the setting $L = \Gamma_d^{(n)}$, where $1 \leq n, d \in \mathbb{Z}$, the collection $\left\{\mathscr{C}_{n,d}^{(\varphi)}\right\}_{(n,d)}$ is *"norm coherent" in the sense of Coleman*, whose definition will be made precise in Section 6. Namely, we shall prove that :

Theorem 1.2. *The square*

$$\begin{array}{ccc} \nabla^{(\varphi)}_{\Gamma^{(n)}_d/K} & \xrightarrow[\sim]{\mathscr{C}^{(\varphi)}_{n,d}} & \mathscr{G}^{(\varphi)}_{\Gamma^{(n)}_d/K} \\ {\scriptstyle \mathscr{N}^{\mathrm{Fesenko/Coleman}}_{(n,d)/(n',d')}}\downarrow & & \downarrow{\scriptstyle \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')}} \\ \nabla^{(\varphi)}_{\Gamma^{(n')}_{d'}/K} & \xrightarrow[\sim]{\mathscr{C}^{(\varphi)}_{n',d'}} & \mathscr{G}^{(\varphi)}_{\Gamma^{(n')}_{d'}/K} \end{array} \tag{1.5}$$

is commutative for every ordered-pair of positive integers (n, d) *and* (n', d') *satisfying* $n' \leq n$ *and* $d' \mid d$.

The vertical arrows in the diagram (1.5) are the "Fesenko theoretic" and the "Laubie theoretic" avatars of the Coleman norm map. In Section 7, we shall observe that the arrow (1.1) is *functorial with respect to the (compatible) base change* (look at [11] for the definition of compatible extensions of K). That is, we shall prove the following theorem.

Theorem 1.3. *If* F/K *is any finite* φ^d*-compatible extension, then the "base change" square*

$$\begin{array}{ccc} \nabla^{(\varphi^d)}_{F} & \xrightarrow{\mathscr{C}^{(\varphi^d)}_{F}} & \mathscr{G}^{(\varphi^d)}_{F} \\ {\scriptstyle \mathscr{N}^{\mathrm{Fesenko},\infty}_{F/K}}\downarrow & & \downarrow{\scriptstyle \mathscr{N}^{\mathrm{Laubie},\infty}_{F/K}} \\ \nabla^{(\varphi)}_{K} & \xrightarrow{\mathscr{C}^{(\varphi)}_{K}} & \mathscr{G}^{(\varphi)}_{K} \end{array} \tag{1.6}$$

is commutative.

The left and right vertical arrows in the diagram (1.6), which will be recalled in Section 7, are defined by the *"existence theorems" in the sense of Fesenko and in the sense of Laubie* respectively. Finally, in Section 8, we shall prove that the arrow (1.1) *respects the "refined filtrations" of* $\nabla^{(\varphi)}_K$ *and of* $\mathscr{G}^{(\varphi)}_K$. More precisely, we have the following theorem.

Theorem 1.4. *Let* $\underline{i} = (i_{(n,d)})$ *be an increasing net in* $\mathbb{Z}_{\geq 0}$ *indexed over the directed set* $(\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}, \preceq)$*, where* $(n', d') \preceq (n, d)$ *if* $n' \leq n$ *and* $d' \mid d$. *The comparison isomorphism* $\mathscr{C}^{(\varphi)}_K : \nabla^{(\varphi)}_K \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_K$ *with respect to* K *restricts to*

$$\mathscr{C}^{(\varphi)}_K : {}_1\left(\nabla^{(\varphi)}_K\right)^{\underline{\varphi}(\underline{i})} \xrightarrow{\sim} \left(\mathscr{G}^{(\varphi)}_K\right)^{\underline{\varphi}^{\mathrm{Laubie}}(\underline{i})},$$

where $\underline{\varphi}(\underline{i})$ *and* $\underline{\varphi}^{\mathrm{Laubie}}(\underline{i})$ *denote the nets* $\left(\varphi_{\Gamma^{(n)}_d/K}(i_{(n,d)})\right)$ *and* $\left(\varphi^{\mathrm{Laubie}}_{\Gamma^{(n)}_d/K}(i_{(n,d)})\right)$ *corresponding to the net* $\underline{i}$ *respectively.*

For certain technical reasons (look at [3] and [9]), we impose the following assumption[1] on the base local field K under consideration.

Assumption 1.5. All through the text, we shall assume that K is a local field with finite residue-class field κ_K of $q = p^f$ elements, and

$$\boldsymbol{\mu}_p(K^{sep}) \subset K^\times,$$

[1] In fact, it is possible to drop this assumption on K. For details, look at [9]

where $\boldsymbol{\mu}_p(K^{sep})$ denotes the multiplicative group of pth roots of unity in K^{sep}.

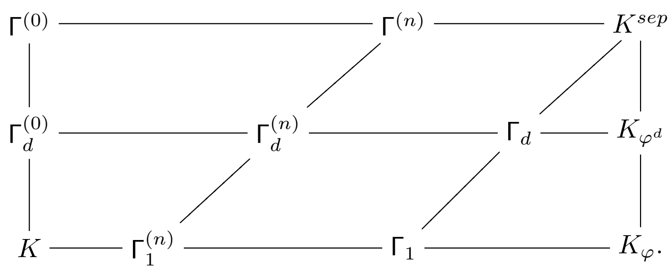
2. The extensions $\Gamma_d^{(n)}/K$ for $1 \leq n, d \in \mathbb{Z}$

The main references that we follow closely in this section are [9, 12]. For $1 \leq d \in \mathbb{Z}$, let K_{φ^d} denote the fixed-field of $\varphi^d \in G_K$. Note that, $K_{\varphi^d} \cap K^{nr} = K_d^{nr}$, where K_d^{nr} denotes the unique unramified extension of K of degree d inside K^{sep} (fixed once and for all throughout the text).

For every $1 \leq n, d \in \mathbb{Z}$, let $\Gamma_d^{(n)}$ be a Galois extension of K, which is the unique maximal n-abelian extension of K_d^{nr} inside K_{φ^d}. Moreover, we set $\Gamma_d^{(0)} := K_d^{nr}$, and also introduce the field extensions over K defined by $\Gamma_d := \bigcup_{0 \leq n \in \mathbb{Z}} \Gamma_d^{(n)}$ for each $1 \leq d \in \mathbb{Z}$, and $\Gamma^{(n)} := \bigcup_{1 \leq d \in \mathbb{Z}} \Gamma_d^{(n)}$ for each $0 \leq n \in \mathbb{Z}$. Therefore, in particular $\Gamma^{(0)} = K^{nr}$. Note that, the collection of field extensions $\left\{\Gamma_d^{(n)}\right\}_{n,d}$ over K has the following basic properties: For each $1 \leq n, n', d, d' \in \mathbb{Z}$,

- Γ_d is the maximal Galois extension over K fixed under φ^d;
- $\Gamma^{(n)} = (\Gamma^{(0)})^{n-ab}$ the "maximal n-abelian" extension of $\Gamma^{(0)}$;
- $\Gamma_{d'}^{(n')} \subseteq \Gamma_d^{(n)}$ if and only if $n' \leq n$ and $d' \mid d$;
- For each pair (n, d) of positive integers, $\Gamma_d^{(n)}$ is an APF-extension over K;
- $\bigcup_{1 \leq n, d \in \mathbb{Z}} \Gamma_d^{(n)} = K^{sep}$, which shows that the fields $\Gamma_d^{(n)}$ for each $0 \leq n \in \mathbb{Z}$ and $1 \leq d \in \mathbb{Z}$ are the "building blocks" of K^{sep}.

Moreover, for each $0 \leq n \in \mathbb{Z}$ and $1 \leq d \in \mathbb{Z}$, the fields $\Gamma_d^{(n)}$, Γ_d and $\Gamma^{(n)}$ sit in the following diagram of Laubie [12]:



3. Generalized Fesenko reciprocity map $\mathbf{\Phi}_{L/K}^{(\varphi)}$ for the extension L/K

Let L/K be an infinite APF-Galois extension with residue degree $[\kappa_L : \kappa_K] =: d_L = d$ satisfiying $K \subset L \subset K_{\varphi^d}$ (i.e., L/K *is a* φ^d*-compatible extension* in the sense of [11]). As usual, the field of norms corresponding to L/K is denoted by $\mathbb{X}(L/K)$ and the completion of the maximal unramified extension $\mathbb{X}(L/K)^{nr}$ of $\mathbb{X}(L/K)$ by $\widetilde{\mathbb{X}}(L/K)$.

In this section, following closely [9], we shall review the construction of the bijective 1-cocycle, called the *generalized Fesenko reciprocity map*

$$\mathbf{\Phi}_{L/K}^{(\varphi)} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \nabla_{L/K}^{(\varphi)} \tag{3.1}$$

for the extension L/K, where the target $\nabla^{(\varphi)}_{L/K}$ is defined explicitly by

$$\nabla^{(\varphi)}_{L/K} := K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{3.2}$$

where $L_0 := L_0^{(K)} = L \cap K^{nr} = K_d^{nr}$, and for the extension L/K as above, the *"diamond subgroup"* $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ of the group $U_{\widetilde{\mathbb{X}}(L/K)}$ of units in the ring of integers of the local field $\widetilde{\mathbb{X}}(L/K)$ is defined by

$$U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} = \mathrm{Pr}^{-1}_{\widetilde{K}}(U_{L_0}),$$

where $\mathrm{Pr}_{\widetilde{K}} : U_{\widetilde{\mathbb{X}}(L/K)} \to U_{\widetilde{K}}$ denotes the continuous projection map on the $\widetilde{K} = \widetilde{L}_0$-coordinate of $U_{\widetilde{\mathbb{X}}(L/K)}$ (for details, [4], [5, 6] and [13]). For the definition of the subgroup Y_{L/L_0} of $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ satisfying the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$, whose definition is involved, we refer the reader to [3] and for more details, to the papers [1, 2, 3] and [7].

The generalized Fesenko reciprocity map $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ for the extension L/K is defined by the commutative triangle

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\boldsymbol{\phi}^{(\varphi)}_{L/K}} & K^\times/N_{L_0/K}{L_0}^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \\ & \searrow^{\boldsymbol{\Phi}^{(\varphi)}_{L/K}} & \Big\downarrow \left(\mathrm{id}_{K^\times/N_{L_0/K}{L_0}^\times}, c_{L/L_0}\right) \\ & & K^\times/N_{L_0/K}{L_0}^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \end{array} \tag{3.3}$$

where

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}{L_0}^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \tag{3.4}$$

is an injective 1-cocycle called, following [8], the *generalized arrow for the extension* L/K, and defined by

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma) = \left(\pi_K^m N_{L_0/K}L_0^\times, \phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right), \tag{3.5}$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $0 \leq m \in \mathbb{Z}$ is the integer satisfying $\sigma\mid_{L_0} = \varphi^m\mid_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$, $\pi_K = \pi_{L_o}$ is the unique prime element of K which is a universal norm in K_{φ^d}/K, and for any $\tau \in \mathrm{Gal}(L/L_0)$, the value $\phi^{(\varphi^d)}_{L/L_0}(\tau)$ of the arrow for the extension L/L_0 at τ is defined by [1, 2, 3] and by [7], which we shall recall now. The arrow

$$\phi^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/U_{\mathbb{X}(L/L_0)} \tag{3.6}$$

for the extension L/L_0 is defined, for each $\tau \in \mathrm{Gal}(L/L_0)$, by the equality

$$\phi^{(\varphi^d)}_{L/L_0}(\tau) = U_\tau . U_{\mathbb{X}(L/L_0)}, \tag{3.7}$$

where $U_\tau \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ is the *unique* solution of the *Dwork's equation*

$$U^{1-\varphi^d} = \Pi^{\tau-1}_{\varphi^d;L/L_0} \tag{3.8}$$

modulo $U_{\mathbb{X}(L/L_0)}$, where $\Pi_{\varphi^d;L/L_0}$ denotes the canonical prime element of the local field $\mathbb{X}(L/L_0)$ defined in Lemma 1.2 and Lemma 1.3 of [8]. In the commutative triangle (3.3), the arrow

$$c_{L/L_0} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \tag{3.9}$$

defined with respect to L/L_0 is the canonical map defined by the inclusion $U_{\mathbb{X}(L/L_0)} \subseteq Y_{L/L_0}$. Recall that (cf. [1, 2, 3] and [7]), the composition $c_{L/L_0} \circ \phi^{(\varphi^d)}_{L/L_0} = \Phi^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is the Fesenko reciprocity map for the extension L/L_0. Thus, for $\sigma \in \mathrm{Gal}(L/K)$, the value $\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma)$ of the generalized Fesenko reciprocity map for the extension L/K is defined by

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma) = \left(\pi_K^m N_{L_0/K}L_0^\times, \Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right), \tag{3.10}$$

where $0 \leq m \in \mathbb{Z}$ is the integer satisfying $\sigma\mid_{L_0} = \varphi^m \mid_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. Define a law of composition $*$ on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}).(\overline{b}, \overline{V})^{(\boldsymbol{\phi}^{(\varphi)}_{L/K})^{-1}((\overline{a},\overline{U}))} \tag{3.11}$$

for every $\overline{a} = a.N_{L_0/K}L_0^\times, \overline{b} = b.N_{L_0/K}L_0^\times \in K^\times/N_{L_0/K}L_0^\times$ with $a, b \in K^\times$ and $\overline{U} = U.U_{\mathbb{X}(L/K)}, \overline{V} = V.U_{\mathbb{X}(L/K)} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ with $U, V \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ is defined by $(\overline{b}, \overline{V})^\sigma = \left(\overline{b}, \overline{V}^{\varphi^{-m}\sigma}\right)$. Then $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ is a topological group under $*$, and the map $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right), \tag{3.12}$$

where the topological group structure on $\mathrm{im}\left(\boldsymbol{\phi}^{(\varphi)}_{L/K}\right)$ is defined with respect to the binary operation $*$ defined by eq. (3.11). Likewise, define a law of composition, again denoted by $*$, on $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ by

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}).(\overline{b}, \overline{V})^{(\boldsymbol{\Phi}^{(\varphi)}_{L/K})^{-1}((\overline{a},\overline{U}))} \tag{3.13}$$

for every $\overline{a} = a.N_{L_0/K}L_0^\times, \overline{b} = b.N_{L_0/K}L_0^\times \in K^\times/N_{L_0/K}L_0^\times$ with $a, b \in K^\times$ and $\overline{U} = U.Y_{L/L_0}, \overline{V} = V.Y_{L/L_0} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ with $U, V \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where the action of $\mathrm{Gal}(L/K)$ on $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined by $(\overline{b}, \overline{V})^\sigma = \left(\overline{b}, \overline{V}^{\varphi^{-m}\sigma}\right)$. Then $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is a topological group under $*$, and the map $\boldsymbol{\Phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{3.14}$$

where the topological group structure on $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined with respect to the binary operation $*$ defined by eq. (3.13).

4. Laubie reciprocity map $\mathbf{\Lambda}_{L/K}^{(\varphi)}$ for the extension L/K

Let L be an APF-extension over the local field K. Then, there exists an equivalence of categories

$$\mathbb{X}_{L/K} : \mathscr{E}_L^{sep} \xrightarrow{\sim} \mathscr{E}_{\mathbb{X}(L/K)}^{sep}, \tag{4.1}$$

where for any field F, we denote by $\mathscr{E}_F^{sep}$ the category whose objects E are the separable extensions of F in a fixed separable closure F^{sep} of F and whose morphisms are the F-embeddings (for details, look at [4]). Moreover, the equivalence $\mathbb{X}_{L/K}$:

- sends the object L of $\mathscr{E}_L^{sep}$ to the object $\mathbb{X}(L/K)$ of $\mathscr{E}_{\mathbb{X}(L/K)}^{sep}$;
- sends the object K^{sep} of $\mathscr{E}_L^{sep}$ to the object $\mathbb{X}(L/K)^{sep}$ of $\mathscr{E}_{\mathbb{X}(L/K)}^{sep}$;
- sends the object $L^{nr} = K^{nr}L$ of $\mathscr{E}_L^{sep}$ to the object $\mathbb{X}(L/K)^{nr}$ of $\mathscr{E}_{\mathbb{X}(L/K)}^{sep}$.
- For each object M of $\mathscr{E}_L^{sep}$, the equivalence $\mathbb{X}_{L/K}$ induces a group isomorphism
$$\mathbb{X}_{L/K} : \mathrm{Aut}_{\mathscr{S}}(M/L) \xrightarrow{\sim} \mathrm{Aut}_{\mathbb{X}_{L/K}(\mathscr{S})}\left(\mathbb{X}_{L/K}(M)/\mathbb{X}(L/K)\right),$$
where $\mathscr{S}$ denotes any subgroup of $\mathrm{Aut}(L)$. Here, $\mathrm{Aut}_{\mathscr{S}}(M/L)$ is the group defined by
$$\mathrm{Aut}_{\mathscr{S}}(M/L) := \{\delta \in \mathrm{Aut}(M) \colon \delta\,|_L \in \mathscr{S},\ \delta(K) = K\}$$
and $\mathbb{X}_{L/K}(\mathscr{S})$ is the subgroup of $\mathrm{Aut}\left(\mathbb{X}(L/K)\right)$ corresponding to the subgroup $\mathscr{S}$ of $\mathrm{Aut}(L)$;
- in particular, setting $\mathscr{S} = \langle \mathrm{id}_L \rangle$ and assuming that the object M of $\mathscr{E}_L^{sep}$ is Galois over L, then clearly $\mathbb{X}_{L/K}(M)$ is Galois over $\mathbb{X}(L/K)$ and the isomorphism
$$\mathbb{X}_{L/K} : \mathrm{Gal}(M/L) \xrightarrow{\sim} \mathrm{Gal}\left(\mathbb{X}_{L/K}(M)/\mathbb{X}(L/K)\right)$$
is compatible with the ramification filtration.
- If L/K is furthermore φ^d-compatible, then the Lubin-Tate splitting $\varphi_L = \varphi^d$ over L is mapped to the Lubin-Tate splitting $\varphi_{\mathbb{X}(L/K)} = \mathbb{X}_{L/K}(\varphi_L)$ over $\mathbb{X}(L/K)$. Moreover, the Lubin-Tate labelling $\mathscr{L}_{\varphi_L}$ is mapped to the Lubin-Tate labelling $\mathscr{L}_{\varphi_{\mathbb{X}(L/K)}}$.

Now, following closely [12], let L be an infinite φ^d-compatible APF-Galois extension over K. Denote the residue-class degree $f_{L/K}$ of the extension L/K by $d_L = d$. It is well-known that, there exists a G_K-equivariant commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{X}}(L/K)^{sep} & \xrightarrow{\ \sim\ } & \overline{\mathbb{F}}_q((\pi_L))^{sep} \\ \text{inc.}\uparrow & & \uparrow\text{inc.} \\ \widetilde{\mathbb{X}}(L/K) & \xrightarrow{\ \sim\ } & \overline{\mathbb{F}}_q((\pi_L)) \\ \text{inc.}\uparrow & & \uparrow\text{inc.} \\ \mathbb{X}(L/K) & \xrightarrow{\ \sim\ } & \mathbb{F}_{q_L}((\pi_L)) \end{array} \tag{4.2}$$

(for details, look at [4, 5, 6, 13]), where π_L is the prime of L chosen from the *Lubin-Tate labelling* $\mathscr{L}_\varphi$ over K. Namely, each $\sigma \in G_K$ defines a unique automorphism

$$\widetilde{\sigma}_M : \widetilde{M} \to \widetilde{M},$$

where M is any separable extension of L, as $\widetilde{M} = \widetilde{K}M$. Clearly, $\widetilde{\sigma}_M(\widetilde{K}) = \widetilde{K}$ and $\widetilde{\sigma}_M \in \mathrm{Aut}_{\mathscr{S}}(\widetilde{M}/\widetilde{L})$, where $\mathscr{S} = \mathrm{Aut}(\widetilde{L})$. Therefore, there exists an automorphism

$$\widetilde{\mathbb{X}}_{L/K}(\widetilde{\sigma}_M) : \widetilde{\mathbb{X}}_{L/K}(\widetilde{M}) \to \widetilde{\mathbb{X}}_{L/K}(\widetilde{M}) \tag{4.3}$$

of $\widetilde{\mathbb{X}}_{L/K}(\widetilde{M})$ whose restriction to $\widetilde{\mathbb{X}}(L/K)$ is the automorphism

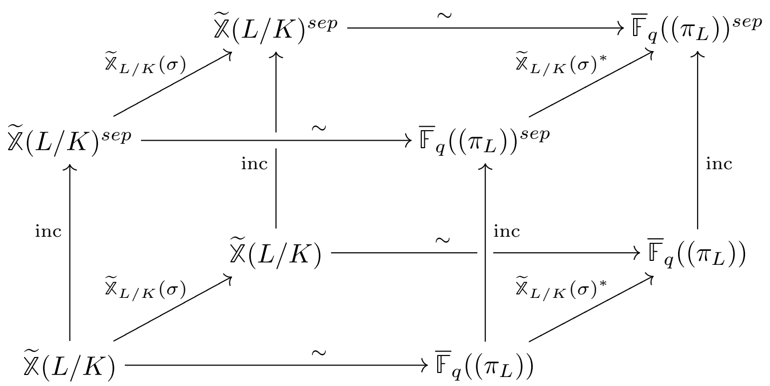
$$\widetilde{\mathbb{X}}_{L/K}(\sigma) = \widetilde{\mathbb{X}}_{L/K}(\widetilde{\sigma}_M) \mid_{\widetilde{\mathbb{X}}(L/K)} : \widetilde{\mathbb{X}}(L/K) \to \widetilde{\mathbb{X}}(L/K) \tag{4.4}$$

of $\widetilde{\mathbb{X}}(L/K)$, which is uniquely determined by the pair $(\nu, \sigma_{L/K}(X)) \in \widehat{\mathbb{Z}} \times X\mathbb{F}_{q^d}[[X]]$, where $\nu \in \widehat{\mathbb{Z}}$ is defined by $\sigma \mid_{K^{nr}} = \mathsf{Frob}_K^{\nu}$ and $\sigma_{L/K}(X) \in X\mathbb{F}_{q^d}[[X]]$ by the equality

$$\left(\sum_i \alpha_i \pi_L^i\right)^{\widetilde{\mathbb{X}}_{L/K}(\sigma)} = \sum_i \mathsf{Frob}_q^{\nu}(\alpha_i)\sigma_{L/K}(\pi_L)^i, \tag{4.5}$$

for each $\sum_i \alpha_i \pi_L^i \in \overline{\mathbb{F}}_q((\pi_L)) \xrightarrow{\sim} \widetilde{\mathbb{X}}(L/K)$ so that the following cube

(4.6)



commutes. As $\mathrm{Gal}(L^{nr}/K) = \mathrm{Aut}_{\mathscr{S}}(L^{nr}/L)$, where $\mathscr{S} = \mathrm{Gal}(L/K)$, it follows that

$$\mathbb{X}_{L/K} : \mathrm{Gal}(L^{nr}/K) \xrightarrow{\sim} \mathrm{Aut}_{\mathbb{X}_{L/K}(\mathscr{S})}(\mathbb{X}(L/K)^{nr}/\mathbb{X}(L/K)). \tag{4.7}$$

Therefore, $\mathrm{Gal}(L^{nr}/K)$ is isomorphic to the group

$$\widetilde{\mathscr{G}}^{(\varphi)}_{L/K} := \left\{(\nu, \sigma_L(X))) : \nu \in \widehat{\mathbb{Z}},\ \sigma_{L/K}(X) \in X\mathbb{F}_{q^d}[[X]]\right\},$$

which is defined under the law of composition

$$\left(\mu, \tau_{L/K}(X)\right)\left(\nu, \sigma_{L/K}(X)\right) = \left(\mu + \nu, (\sigma_{L/K}^{\varphi^{\mu}} \circ \tau_{L/K})(X)\right)$$

for every $(\mu, \tau_{L/K}(X)), (\nu, \sigma_{L/K}(X)) \in \widetilde{\mathscr{G}}^{(\varphi)}_{L/K}$, where the isomorphism

$$\widetilde{\mathbf{\Lambda}}^{(\varphi)}_{L/K} : \mathrm{Gal}(L^{nr}/K) \xrightarrow{\sim} \widetilde{\mathscr{G}}^{(\varphi)}_{L/K} \tag{4.8}$$

is defined by

$$\sigma \mapsto (\nu, \sigma_{L/K}(X)), \tag{4.9}$$

for each $\sigma \in \mathrm{Gal}(L^{nr}/K)$, where $\nu \in \widehat{\mathbb{Z}}$ satisfies $\sigma \mid_{K^{nr}} = \mathsf{Frob}_K^{\nu}$ and $\sigma_{L/K}(X) \in X\mathbb{F}_{q^d}[[X]]$ is defined by the morphism $\widetilde{\mathbb{X}}_{L/K}(\sigma)$ introduced in (4.4) whose action on

$\overline{\mathbb{F}}_q((\pi_L)) \xrightarrow{\sim} \widetilde{\mathbb{X}}(L/K)$ is described in (4.5). Moreover, the surjective homomorphism

$$\sqcap_1 : \widetilde{\mathscr{G}}^{(\varphi)}_{L/K} \to \mathscr{G}^{(\varphi)}_{L/K}, \tag{4.10}$$

where

$$\mathscr{G}^{(\varphi)}_{L/K} := \left\{ (\sqcap_d(\nu), \sigma_{L/K}(X)) \in \mathbb{Z}/d\mathbb{Z} \times X\mathbb{F}_{q^d}[[X]] : (\nu, \sigma_{L/K}(X)) \in \widetilde{\mathscr{G}}^{(\varphi)}_{L/K} \right\}, \tag{4.11}$$

is defined by

$$\sqcap_1 : (\nu, \sigma_{L/K}(X)) \mapsto \left(\sqcap_d(\nu), \sigma_{L/K}(X) \right), \tag{4.12}$$

for every $(\nu, \sigma_{L/K}(X)) \in \widetilde{\mathscr{G}}^{(\varphi)}_{L/K}$. Here, $\sqcap_d : \widehat{\mathbb{Z}} \to \mathbb{Z}/d\mathbb{Z}$ given by $\sqcap_d : \nu \mapsto \nu_d$, for every $\nu \in \widehat{\mathbb{Z}}$, denotes the natural projection on the d-th coordinate. Then, $\sqcap_1$ induces an isomorphism

$$\mathbf{\Lambda}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{L/K}, \tag{4.13}$$

defined by the composition

$$\mathrm{Gal}(L^{nr}/K) \xrightarrow[\sim]{\widetilde{\mathbf{\Lambda}}^{(\varphi)}_{L/K}} \widetilde{\mathscr{G}}^{(\varphi)}_{L/K} \xrightarrow{\sqcap_1} \mathscr{G}^{(\varphi)}_{L/K}, \tag{4.14}$$

as

$$\widetilde{\mathbf{\Lambda}}^{(\varphi)}_{L/K} \mid_{\mathrm{Gal}(L^{nr}/L)} : \mathrm{Gal}(L^{nr}/L) \xrightarrow{\sim} \ker(\sqcap_1). \tag{4.15}$$

The topological isomorphism (4.13) defined by (4.14) is called the *Laubie reciprocity law for the extension* L/K. For more details, see [12].

5. Relationship between $\mathbf{\Phi}^{(\varphi)}_{L/K}$ and $\mathbf{\Lambda}^{(\varphi)}_{L/K}$ for the extension L/K

The aim of this section is to prove Theorem 1.1 stated in Section 1. As in Sections 4 and 3, let L be an infinite φ^d-compatible APF-Galois extension over K. We shall construct a topological isomorphism $\mathscr{C}^{(\varphi)}_{L/K} : \nabla^{(\varphi)}_{L/K} \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{L/K}$ so that, the local reciprocity law $\mathbf{\Lambda}^{(\varphi)}_{L/K}$ of Laubie for L/K can be expressed in terms of the generalized Fesenko reciprocity law $\mathbf{\Phi}^{(\varphi)}_{L/K}$ for L/K and the comparison isomorphism $\mathscr{C}^{(\varphi)}_{L/K}$ with respect to L/K.

Note that, there exists an isomorphism

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L_0/K) \times \mathrm{Gal}(L/L_0) \tag{5.1}$$

defined by

$$\sigma \mapsto \left(\sigma \mid_{L_0}, \varphi^{-m}\sigma \right), \tag{5.2}$$

where $\sigma \mid_{L_0} = \varphi^m \mid_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. By abelian local class field theory, there exists the Artin reciprocity isomorphism $\mathsf{Art}_{L_0/K} : \mathrm{Gal}(L_0/K) \xrightarrow{\sim} K^\times/\mathrm{N}_{L_0/K}L_0^\times$. On the other hand, $\xi_{L_0/K} : \mathrm{Gal}(L_0/K) \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}$ as $L_0 = K_d^{nr}$. Therefore, to study the structure of the Galois group $\mathrm{Gal}(L/K)$ corresponding to L/K, it suffices to focus on $\mathrm{Gal}(L/L_0)$ corresponding to the extension L/L_0, which is indeed the underlying idea of [8].

By the isomorphism (5.1) defined by (5.2) and following the basic idea of the previous paragraph, to construct the comparison isomorphism (1.3) with respect to L/K, it therefore suffices to consider the extension L/L_0, which is an infinite totally-ramified APF-Galois extension satisfying $L_0 \subset L \subset (L_0)_{\varphi^d}$, where $\varphi_{L_0} = \varphi^d$ is the Lubin-Tate splitting over L_0 that we shall fix all through this section, and construct the comparison isomorphism $\mathscr{C}^{(\varphi^d)}_{L/L_0} : \nabla^{(\varphi^d)}_{L/L_0} \xrightarrow{\sim} \mathscr{G}^{(\varphi^d)}_{L/L_0}$ with respect to the extension L/L_0, which satisfies $\mathscr{C}^{(\varphi^d)}_{L/L_0} \circ \mathbf{\Phi}^{(\varphi^d)}_{L/L_0} = \mathbf{\Lambda}^{(\varphi^d)}_{L/L_0}$. In fact, (3.2) in this setting, specializes to

$$\nabla^{(\varphi^d)}_{L/L_0} = \left\langle \overline{1}_{L_0} \right\rangle \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/Y_{L/L_0}, \tag{5.3}$$

which is isomorphic to $U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/Y_{L/L_0}$ under the projection map on the 2nd-coordinate

$$\Pi_2 : \nabla^{(\varphi^d)}_{L/L_0} \xrightarrow{\sim} U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/Y_{L/L_0}, \tag{5.4}$$

and the generalized Fesenko reciprocity map $\mathbf{\Phi}^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \xrightarrow{\sim} \nabla^{(\varphi^d)}_{L/L_0}$ degenerates to the Fesenko reciprocity map $\Phi^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \xrightarrow{\sim} U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/Y_{L/L_0}$ under the commutative diagram

$$\begin{array}{ccc} & & \nabla^{(\varphi^d)}_{L/L_0} \\ & \overset{\mathbf{\Phi}^{(\varphi^d)}_{L/L_0}}{\underset{\sim}{\nearrow}} & \\ \mathrm{Gal}(L/L_0) & & \downarrow \Pi_2 \\ & \underset{\Phi^{(\varphi^d)}_{L/L_0}}{\overset{\sim}{\searrow}} & \\ & & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/Y_{L/L_0}. \end{array} \tag{5.5}$$

Also, (4.11) specializes to

$$\mathscr{G}^{(\varphi^d)}_{L/L_0} = \left\{ (0_{\mathbb{Z}/d\mathbb{Z}}, \sigma_{L/L_0}(X)) \in \left\langle 0_{\mathbb{Z}/d\mathbb{Z}} \right\rangle \times X\mathbb{F}_{q^d}[[X]] : (0_{\widehat{\mathbb{Z}}}, \sigma_{L/L_0}(X)) \in \widetilde{\mathscr{G}}^{(\varphi^d)}_{L/L_0} \right\}, \tag{5.6}$$

where

$$\widetilde{\mathscr{G}}^{(\varphi^d)}_{L/L_0} = \left\langle 0_{\widehat{\mathbb{Z}}} \right\rangle \times X\mathbb{F}_{q^d}[[X]]. \tag{5.7}$$

Therefore, the topological isomorphism

$$\Upsilon^{(\varphi)}_{L/K} : \nabla^{(\varphi)}_{L/K} \xrightarrow{\sim} K^\times/\mathrm{N}_{L_0/K}L_0^\times \times \nabla^{(\varphi^d)}_{L/L_0}, \tag{5.8}$$

defined by

$$\Upsilon^{(\varphi)}_{L/K} : (\overline{a}, \overline{U}) \mapsto \left(\overline{a}, (\overline{1}_{L_0}, \overline{U})\right), \tag{5.9}$$

for every $\overline{a} = a.\mathrm{N}_{L_0/K}{L_0}^\times$ with $a \in K^\times$ and $\overline{U} = U.Y_{L/L_0}$ with $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}$, and the topological isomorphism

$$\Xi^{(\varphi)}_{L/K} : \mathscr{G}^{(\varphi)}_{L/K} \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z} \times \mathscr{G}^{(\varphi^d)}_{L/L_0}, \tag{5.10}$$

defined by

$$\Xi^{(\varphi)}_{L/K} : \left(\Pi_d(\nu), \sigma_{L/K}(X)\right) \mapsto \left(\Pi_d(\nu), \left(0_{\mathbb{Z}/d\mathbb{Z}}, (\varphi^{-\nu}\sigma)_{L/L_0}(X)\right)\right), \tag{5.11}$$

for every $(\nu, \sigma_{L/K}(X)) \in \widetilde{\mathscr{G}}_{L/K}^{(\varphi)}$, induce the comparison isomorphism
(5.12)

$$\mathscr{C}_{L/K}^{(\varphi)} : \nabla_{L/K}^{(\varphi)} \xrightarrow[\sim]{\Upsilon_{L/K}^{(\varphi)}} K^\times/\mathrm{N}_{L_0/K}L_0^\times \times \nabla_{L/L_0}^{(\varphi^d)} \xrightarrow[\sim]{\left(\xi_{L_0/K}\circ \mathsf{Art}_{L_0/K}^{-1}, \mathscr{C}_{L/L_0}^{(\varphi^d)}\right)} \mathbb{Z}/d\mathbb{Z} \times \mathscr{G}_{L/L_0}^{(\varphi^d)} \xrightarrow[\sim]{(\Xi_{L/K}^{(\varphi)})^{-1}} \mathscr{G}_{L/K}^{(\varphi)}$$

with respect to the extension L/K satisfying the identity $\mathscr{C}_{L/K}^{(\varphi)} \circ \mathbf{\Phi}_{L/K}^{(\varphi)} = \mathbf{\Lambda}_{L/K}^{(\varphi)}$. In fact, it is clear that the arrow $\mathscr{C}_{L/K}^{(\varphi)}$ is a topological isomorphism, as all the arrows in the definition of $\mathscr{C}_{L/K}^{(\varphi)}$ are topological isomorphisms. Now, it remains to check the identity relating the local reciprocity laws $\mathbf{\Phi}_{L/K}^{(\varphi)}$ and $\mathbf{\Lambda}_{L/K}^{(\varphi)}$. In order to do so, let $\sigma \in \mathrm{Gal}(L/K)$. Assume that, $0 \leq m \in \mathbb{Z}$ satisfying $\sigma\mid_{L_0} = \varphi^m\mid_{L_0} \in \mathrm{Gal}(L_0/K)$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. Then, by (3.10), the commutative triangle (5.5), and by (5.12),

$$\begin{aligned}
\mathscr{C}_{L/K}^{(\varphi)} \circ \mathbf{\Phi}_{L/K}^{(\varphi)}(\sigma) &= \mathscr{C}_{L/K}^{(\varphi)}\left(\pi_K^m \mathrm{N}_{L_0/K}L_0^\times, \Phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right) \\
&= \mathscr{C}_{L/K}^{(\varphi)}\left(\pi_K^m \mathrm{N}_{L_0/K}L_0^\times, \sqcap_2 \circ \mathbf{\Phi}_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right) \\
&= (\Xi_{L/K}^{(\varphi)})^{-1} \circ \left(\xi_{L_0/K}\circ \mathsf{Art}_{L_0/K}^{-1}, \mathscr{C}_{L/L_0}^{(\varphi^d)}\right) \circ \Upsilon_{L/K}^{(\varphi)}\left(\pi_K^m \mathrm{N}_{L_0/K}L_0^\times, \sqcap_2 \circ \mathbf{\Phi}_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right),
\end{aligned}$$

and by (5.8) and (5.9), together with the definition of the arrow $\xi_{L_0/K}$, the commutative triangle (5.5), and the existence of the comparison isomorphism $\mathscr{C}_{L/L_0}^{(\varphi^d)}$ with respect to the extension L/L_0,

$$\begin{aligned}
\mathscr{C}_{L/K}^{(\varphi)} \circ \mathbf{\Phi}_{L/K}^{(\varphi)}(\sigma) &= (\Xi_{L/K}^{(\varphi)})^{-1} \circ \left(\xi_{L_0/K}\circ \mathsf{Art}_{L_0/K}^{-1}, \mathscr{C}_{L/L_0}^{(\varphi^d)}\right)\left(\pi_K^m \mathrm{N}_{L_0/K}L_0^\times, \left(\overline{1}_{L_0}, \sqcap_2 \circ \mathbf{\Phi}_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right)\right) \\
&= (\Xi_{L/K}^{(\varphi)})^{-1}\left(\xi_{L_0/K}\circ \mathsf{Art}_{L_0/K}^{-1}(\pi_K^m \mathrm{N}_{L_0/K}L_0^\times), \mathscr{C}_{L/L_0}^{(\varphi^d)}\left(\overline{1}_{L_0}, \sqcap_2 \circ \mathbf{\Phi}_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right)\right) \\
&= (\Xi_{L/K}^{(\varphi)})^{-1}\left(m(\mathrm{mod}\ d), \mathscr{C}_{L/L_0}^{(\varphi^d)} \circ \mathbf{\Phi}_{L/L_0}^{(\varphi^d)}\left(\varphi^{-m}\sigma\right)\right) \\
&= (\Xi_{L/K}^{(\varphi)})^{-1}\left(m(\mathrm{mod}\ d), \mathbf{\Lambda}_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right),
\end{aligned}$$

which, by (5.10), (5.11), and by the definition of the Laubie reciprocity law $\mathbf{\Lambda}_{L/K}^{(\varphi)}$ for the extension L/K, yields the desired identity

$$\begin{aligned}
\mathscr{C}_{L/K}^{(\varphi)} \circ \mathbf{\Phi}_{L/K}^{(\varphi)}(\sigma) &= (\Xi_{L/K}^{(\varphi)})^{-1}\big(m(\mathrm{mod}\ d), \big(0_{\mathbb{Z}/d\mathbb{Z}}, (\varphi^{-m}\sigma)_{L/L_0}(X)\big)\big) \\
&= \big(m(\mathrm{mod}\ d), \sigma_{L/K}(X)\big) \\
&= \mathbf{\Lambda}_{L/K}^{(\varphi)}(\sigma).
\end{aligned}$$

Thus, it remains to prove the existence of the comparison isomorphism $\mathscr{C}_{L/L_0}^{(\varphi^d)} : \nabla_{L/L_0}^{(\varphi^d)} \xrightarrow{\sim} \mathscr{G}_{L/L_0}^{(\varphi^d)}$ for the extension L/L_0, which is in fact a corollary to Section 3.7 of [12], which we state now as the following lemma.

Lemma 5.1 (Corollary to Section 3.7 of [12])**.** *Let L be an infinite φ^d-compatible APF-Galois extension over K. Then, there exists a topological isomorphism (the comparison isomorphism with respect to L/L_0)*

$$\mathscr{C}_{L/L_0}^{(\varphi^d)} : \nabla_{L/L_0}^{(\varphi^d)} \xrightarrow{\sim} \mathscr{G}_{L/L_0}^{(\varphi^d)}$$

satisfying

$$\mathscr{C}^{(\varphi^d)}_{L/L_0} \circ \boldsymbol{\Phi}^{(\varphi^d)}_{L/L_0} = \boldsymbol{\Lambda}^{(\varphi^d)}_{L/L_0},$$

where L_0 *denotes the field* $L \cap K^{nr} = K^{nr}_d$.

Proof. The proof follows from Section 3.7 of [12] with K replaced by L_0 and ϕ replaced by φ^d. □

Thus, the proof of Theorem 1.1 is now complete.

6. Norm coherence of $\mathscr{C}^{(\varphi)}_{n,d}$ for $1 \leq n, d \in \mathbb{Z}$

In this section, we shall prove the commutativity of the norm-coherence square (1.5) for every ordered-pair of positive integers (n, d) and (n', d') satisfying $n' \leq n$ and $d' \mid d$, and get Theorem 1.2. This theorem also yields the construction of the comparison isomorphism (1.1) with respect to K satisfying (1.2).

Let $(n, d), (n', d') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying $n' \leq n$ and $d' \mid d$, and consider the extensions $\Gamma^{(n)}_d$ and $\Gamma^{(n')}_{d'}$ over the local field K. Clearly, there exist the inclusions $K \subseteq \Gamma^{(n')}_{d'} \subseteq \Gamma^{(n)}_d$ (look at Section 2).

Following closely [9], then there exists the topological homomorphism, which is the "Fesenko theoretic" avatar of the Coleman norm map

$$\mathscr{N}^{\text{Fesenko/Coleman}}_{(n,d)/(n',d')} : \nabla^{(\varphi)}_{\Gamma^{(n)}_d/K} \to \nabla^{(\varphi)}_{\Gamma^{(n')}_{d'}/K} \tag{6.1}$$

defined by

$$\mathscr{N}^{\text{Fesenko/Coleman}}_{(n,d)/(n',d')} : (\overline{a}, \overline{U}) \mapsto \left(e^{\text{CFT}}_{K^{nr}_d/K^{nr}_{d'}}(\overline{a}), \mathscr{N}^{\text{Coleman}}_{\Gamma^{(n)}_d/\Gamma^{(n')}_{d'}}(\overline{U}) \right), \tag{6.2}$$

for $(\overline{a}, \overline{U}) \in \nabla^{(\varphi)}_{\Gamma^{(n)}_d/K}$ with $a \in K^\times$ and $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma^{(n)}_d/K)}$, where the arrow $e^{\text{CFT}}_{K^{nr}_d/K^{nr}_{d'}} : K^\times/\text{N}_{K^{nr}_d/K}(K^{nr\times}_d) \to K^\times/\text{N}_{K^{nr}_{d'}/K}(K^{nr\times}_{d'})$ is the natural inclusion defined by the existence theorem of local class field theory and $\mathscr{N}^{\text{Coleman}}_{\Gamma^{(n)}_d/\Gamma^{(n')}_{d'}} : U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma^{(n)}_d/K)}/Y_{\Gamma^{(n)}_d/K} \to U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma^{(n')}_{d'}/K)}/Y_{\Gamma^{(n')}_{d'}/K}$ is the Coleman norm map from $\Gamma^{(n)}_d$ to $\Gamma^{(n')}_{d'}$ defined by Lemma 2.21 together with equations (2.47) and (2.48) of [8], so that the following square

$$\begin{array}{ccc} \text{Gal}(\Gamma^{(n)}_d/K) & \xrightarrow{\ \text{res}_{\Gamma^{(n')}_{d'}}\ } & \text{Gal}(\Gamma^{(n')}_{d'}/K) \\ \downarrow{\scriptstyle \boldsymbol{\Phi}^{(\varphi)}_{\Gamma^{(n)}_d/K}} & & \downarrow{\scriptstyle \boldsymbol{\Phi}^{(\varphi)}_{\Gamma^{(n')}_{d'}/K}} \\ \nabla^{(\varphi)}_{\Gamma^{(n)}_d/K} & \xrightarrow{\ \mathscr{N}^{\text{Fesenko/Coleman}}_{(n,d)/(n',d')}\ } & \nabla^{(\varphi)}_{\Gamma^{(n')}_{d'}/K} \end{array} \tag{6.3}$$

commutes.

Also, following closely [12], there exists the topological homomorphism, which is the "Laubie theoretic" avatar of the Coleman norm map

$$\mathscr{N}^{\text{Laubie/Coleman}}_{(n,d)/(n',d')} : \mathscr{G}^{(\varphi)}_{\Gamma^{(n)}_d/K} \to \mathscr{G}^{(\varphi)}_{\Gamma^{(n')}_{d'}/K} \tag{6.4}$$

defined by the commutative diagram

(6.5)
$$\begin{array}{ccc}
\mathrm{Gal}(\Gamma_d^{(n)}/K) & \xrightarrow{\ \mathrm{res}_{\Gamma_{d'}^{(n')}}\ } & \mathrm{Gal}(\Gamma_{d'}^{(n')}/K) \\
\Big\downarrow {\scriptstyle \mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K}} & & \Big\downarrow {\scriptstyle \mathbf{\Lambda}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}} \\
\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K} & \xrightarrow{\ \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')}\ } & \mathscr{G}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K} \\
\Big\downarrow & & \Big\downarrow \\
\mathbb{Z}/d_{\Gamma_d^{(n)}}\mathbb{Z} \times \mathbb{F}_{q^{d_{\Gamma_d^{(n)}}}}[[X]] & \xrightarrow{\ (c^{(n,d)}_{(n',d')}, \vartheta_{(n,d)/(n',d')})\ } & \mathbb{Z}/d_{\Gamma_{d'}^{(n')}}\mathbb{Z} \times \mathbb{F}_{q^{d_{\Gamma_{d'}^{(n')}}}}[[X]],
\end{array}$$

where the 1st coordinate $c^{(n,d)}_{(n',d')} : \mathbb{Z}/d_{\Gamma_d^{(n)}}\mathbb{Z} \to \mathbb{Z}/d_{\Gamma_{d'}^{(n')}}\mathbb{Z}$ of the bottom horizontal arrow is the canonical map and the 2nd coordinate $\vartheta_{(n,d)/(n',d')} : \mathbb{F}_{q^{d_{\Gamma_d^{(n)}}}}((X))^\times \to \mathbb{F}_{q^{d_{\Gamma_{d'}^{(n')}}}}((X))^\times$ of the bottom horizontal arrow is defined by $\vartheta_{(n,d)/(n',d')} : \sum_i \beta_i X^i \mapsto \sum_i \alpha_i X^i$, for $\sum_i \beta_i X^i \in \mathbb{F}_{q^{d_{\Gamma_d^{(n)}}}}((X))^\times$, where $\sum_i \beta_i \pi^i_{\Gamma_d^{(n)}} \mapsto \sum_i \alpha_i \pi^i_{\Gamma_{d'}^{(n')}}$ under the commutative square

(6.6)
$$\begin{array}{ccc}
\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)^\times & \xrightarrow{\ \widetilde{\mathrm{N}}_{\Gamma_d^n/\Gamma_{d'}^{(n')}}\ } & \widetilde{\mathbb{X}}(\Gamma_{d'}^{(n')}/K)^\times \\
\Big\downarrow \wr & & \Big\downarrow \wr \\
\overline{\mathbb{F}}_q((\pi_{\Gamma_d^{(n)}}))^\times & \longrightarrow & \overline{\mathbb{F}}_q((\pi_{\Gamma_{d'}^{(n')}}))^\times.
\end{array}$$

Now, glueing the commutative squares (6.3) and (6.5),

(6.7)
$$\begin{array}{ccccc}
& \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K} & \xrightarrow{\ \mathscr{N}^{\mathrm{Fesenko/Coleman}}_{(n,d)/(n',d')}\ } & \nabla^{(\varphi)}_{\Gamma_{d'}^{(n')}/K} & \\
& \Big\uparrow {\scriptstyle \mathbf{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}} & & \Big\uparrow {\scriptstyle \mathbf{\Phi}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}} & \\
\mathscr{C}^{(\varphi)}_{n,d} = (\mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K}) \circ (\mathbf{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K})^{-1} & \mathrm{Gal}(\Gamma_d^{(n)}/K) & \xrightarrow{\ \mathrm{res}_{\Gamma_{d'}^{(n')}}\ } & \mathrm{Gal}(\Gamma_{d'}^{(n')}/K) & (\mathbf{\Lambda}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}) \circ (\mathbf{\Phi}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K})^{-1} = \mathscr{C}^{(\varphi)}_{n',d'} \\
& \Big\downarrow {\scriptstyle \mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K}} & & \Big\downarrow {\scriptstyle \mathbf{\Lambda}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}} & \\
& \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K} & \xrightarrow{\ \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')}\ } & \mathscr{G}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K} &
\end{array}$$

the commutativity of the square (1.5), for every ordered-pair of positive integers (n,d) and (n',d') satisfying $n' \leq n$ and $d' \mid d$, follows; namely, the norm coherence of the collection of topological isomorphisms $\left\{ \mathscr{C}^{(\varphi)}_{n,d} : \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K} \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K} \right\}_{(n,d)}$ follows.

Therefore, the collection of topological isomorphisms $\left\{\mathscr{C}_{n,d}^{(\varphi)} : \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K} \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right\}_{(n,d)}$ turns out to be an isomorphism

$$\left\{\mathscr{C}_{n,d}^{(\varphi)} : \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K} \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right\}_{(n,d)} : \left\{\nabla^{(\varphi)}_{\Gamma_d^{(n)}/K}; \mathscr{N}^{\text{Fesenko/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n\\ d'\mid d}} \xrightarrow{\sim} \left\{\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}; \mathscr{N}^{\text{Laubie/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n\\ d'\mid d}}$$

between the projective systems $\left\{\nabla^{(\varphi)}_{\Gamma_d^{(n)}/K}; \mathscr{N}^{\text{Fesenko/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n\\ d'\mid d}}$ and $\left\{\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}; \mathscr{N}^{\text{Laubie/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n\\ d'\mid d}}$ (for details, look at Section 6 of [9] and Section Section 1.4 of [12]). Now, passing to the projective limits, there exists an isomorphism, the comparison isomorphism (1.1) with respect to K

$$\mathscr{C}_K^{(\varphi)} = \varprojlim_{(n,d)} \mathscr{C}_{n,d}^{(\varphi)} : \nabla_K^{(\varphi)} = \varprojlim_{(n,d)} \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K} \xrightarrow{\sim} \mathscr{G}_K^{(\varphi)} = \varprojlim_{(n,d)} \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K},$$

which sits in the commutative triangle (1.2). That is, the following diagram

$$\begin{array}{ccc} & & \nabla_K^{(\varphi)} \\ & \overset{\mathbf{\Phi}_K^{(\varphi)}}{\underset{\sim}{\nearrow}} & \\ G_K & & \Big\downarrow \mathscr{C}_K^{(\varphi)} \\ & \underset{\mathbf{\Lambda}_K^{(\varphi)}}{\overset{\sim}{\searrow}} & \\ & & \mathscr{G}_K^{(\varphi)} \end{array}$$

is commutative.

7. Functoriality with respect to the base change

In this section, we shall prove Theorem 1.3. Note that, the comparison isomorphism $\mathscr{C}_K^{(\varphi)} : \nabla_K^{(\varphi)} \xrightarrow{\sim} \mathscr{G}_K^{(\varphi)}$ with respect to K constructed in Section 6 is *functorial with respect to the (compatible) base change.* That is, if F/K is any finite φ^d-compatible extension, then the "base change" square (1.6), namely the square

$$\begin{array}{ccc} \nabla_F^{(\varphi^d)} & \xrightarrow{\mathscr{C}_F^{(\varphi^d)}} & \mathscr{G}_F^{(\varphi^d)} \\ {\scriptstyle \mathscr{N}^{\text{Fesenko},\infty}_{F/K}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathscr{N}^{\text{Laubie},\infty}_{F/K}} \\ \nabla_K^{(\varphi)} & \xrightarrow{\mathscr{C}_K^{(\varphi)}} & \mathscr{G}_K^{(\varphi)} \end{array}$$

is commutative, where the left and right vertical arrows in the diagram (1.6) are defined by (look at the proof of the equation (7.4) in [9] and Section 3.4 in [12]) the commutative diagram

$$\begin{array}{ccccc} \nabla_F^{(\varphi^d)} & \underset{\sim}{\xleftarrow{\mathbf{\Phi}_F^{(\varphi^d)}}} & G_F & \underset{\sim}{\xrightarrow{\mathbf{\Lambda}_F^{(\varphi^d)}}} & \mathscr{G}_F^{(\varphi^d)} \\ {\scriptstyle \mathscr{N}^{\text{Fesenko},\infty}_{F/K}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{inc.}} & & \Big\downarrow{\scriptstyle \mathscr{N}^{\text{Laubie},\infty}_{F/K}} \\ \nabla_K^{(\varphi)} & \underset{\sim}{\xleftarrow{\mathbf{\Phi}_K^{(\varphi)}}} & G_K & \underset{\sim}{\xrightarrow{\mathbf{\Lambda}_K^{(\varphi)}}} & \mathscr{G}_K^{(\varphi)} \end{array} \tag{7.1}$$

which also proves the commutativity of the square (1.6).

8. Ramification theory – revisited

The aim of this section is to prove Theorem 1.4. That is, the comparison isomorphism $\mathscr{C}_K^{(\varphi)} : \nabla_K^{(\varphi)} \xrightarrow{\sim} \mathscr{G}_K^{(\varphi)}$ with respect to K constructed in Section 6 is compatible with the "refined filtration" of $\nabla_K^{(\varphi)}$ introduced in [10], and the "refined filtration" of $\mathscr{G}_K^{(\varphi)}$ that we shall introduce in this section, extending the work of [12]. As in the previous sections, let L be an infinite φ^d-compatible APF-Galois extension over K.

The ramification theorem for the generalized Fesenko reciprocity map $\boldsymbol{\Phi}_{L/K}^{(\varphi)}$ for the extension L/K states (look at [8]), for each $0 \leq w \in \mathbb{Z}$, the inclusion

(8.1)
$$\boldsymbol{\Phi}_{L/K}^{(\varphi)}\left(\mathrm{Gal}(L/K)_w - \mathrm{Gal}(L/K)_{w+1}\right) \subseteq \left\langle 1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times \left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w Y_{L/L_0}/Y_{L/L_0} - Q_{L/L_0}^{w+1} \right),$$

where

$$Q_{L/L_0}^w = c_{L/L_0}\left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \cap \mathrm{im}(\phi_{L/L_0}^{(\varphi^d)}) \right).$$

Consequently, we have the following proposition, which sharpens the results in [8, 10].

Proposition 8.1. *Let L be an infinite φ^d-compatible APF-Galois extension over K. For any $0 \leq w \in \mathbb{Z}$, the equalities*

$$\boldsymbol{\Phi}_{L/K}^{(\varphi)}\left(\mathrm{Gal}(L/K)_w - \mathrm{Gal}(L/K)_{w+1}\right) = \left\langle 1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times \left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w Y_{L/L_0}/Y_{L/L_0} - Q_{L/L_0}^{w+1} \right)$$

and

$$\boldsymbol{\Phi}_{L/K}^{(\varphi)}\left(\mathrm{Gal}(L/K)_w\right) = \left\langle 1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times \left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w Y_{L/L_0}/Y_{L/L_0}.$$

hold.

Proof. Suppose that the inclusion (8.1) were strict. Then, there exists an element ξ in $\left\langle 1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times \left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w Y_{L/L_0}/Y_{L/L_0} - Q_{L/L_0}^{w+1} \right)$ that does not belong to $\boldsymbol{\Phi}_{L/K}^{(\varphi)}\left(\mathrm{Gal}(L/K)_w - \mathrm{Gal}(L/K)_{w+1}\right)$. Therefore, there exists $0 \leq w_o \in \mathbb{Z}$ such that $w_o < w$ and $\xi \in \boldsymbol{\Phi}_{L/K}^{(\varphi)}\left(\mathrm{Gal}(L/K)_{w_o}\right)$. Choose the maximal one among such w_o, and denote it again by w_o. Then, clearly $\xi \in \boldsymbol{\Phi}_{L/K}^{(\varphi)}\left(\mathrm{Gal}(L/K)_{w_o} - \mathrm{Gal}(L/K)_{w_o+1}\right)$, by the maximality of w_o, which implies, by (8.1), that $\xi \in \left\langle 1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times \left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^{w_o} Y_{L/L_0}/Y_{L/L_0} - Q_{L/L_0}^{w_o+1} \right)$. On the other hand, $\xi \in \left\langle 1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times Q_{L/L_0}^{w_o+1}$ as

$$\xi = \left(1_{K^\times/\mathrm{N}_{K_d^{nr}/K} K_d^{nr\times}}, \sqcap_2(\xi) \right)$$

with $\sqcap_2(\xi) \in Q_{L/L_0}^w = c_{L/L_0}\left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \cap \mathrm{im}(\phi_{L/L_0}^{(\varphi^d)}) \right) \subseteq Q_{L/L_0}^{w_o+1}$, by the choice of the element ξ and by (3.10), which leads to a contradiction. Now, the second equality follows from the first one. □

Following closely [10], for $0 \leq w \in \mathbb{R}$, let

$$\left(\nabla_{L/K}^{(\varphi)} \right)^w = \nabla_{L/K}^w = K^\times/N_{L_0/K} L_0^\times \times \left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \right)^w Y_{L/L_0}/Y_{L/L_0}.$$

Now, if M/K is an infinite Galois sub-extension of L/K satisfying $K \subset M \subset K_{\varphi^{d'}}$ with residue class degree $[\kappa_M : \kappa_K] = d'$, where $d' \mid d$, then, for $0 \leq w \in \mathbb{Z}$, the inequality

$$\psi_{M/M_0}(w) \leq \psi_{L/L_0}(w)$$

is satisfied (for details, look at Proposition 4.2 and Remark 4.3 in [10]). Here, for an APF-extension E/K, $\psi_{E/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ denotes the *"Hasse-Herbrand function" for* E/K. The inverse $\psi_{E/K}^{-1}$ of $\psi_{E/K}$ is denoted by $\varphi_{E/K}$. Moreover,

$$\widetilde{\mathscr{N}}_{L/M}^{\text{Coleman}}\left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^{w} Y_{L/L_0}/Y_{L/L_0}\right) \subseteq \left(U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}\right)^{w} Y_{M/M_0}/Y_{M/M_0}, \tag{8.2}$$

for every $0 \leq w \in \mathbb{Z}$. Therefore, for the local fields $K \subset M \subset L$, and for $0 \leq w_{L/K}, w_{M/K} \in \mathbb{R}$ satisfying $w_{M/K} \leq w_{L/K}$, the map $\mathscr{N}_{L/M}^{\text{Fesenko/Coleman}} : \nabla_{L/K}^{(\varphi)} \to \nabla_{M/K}^{(\varphi)}$ restricts to

$$\mathscr{N}_{L/M}^{\text{Fesenko/Coleman}} : \nabla_{L/K}^{\psi_{L/L_0}(w_{L/K})} \to \nabla_{M/K}^{\psi_{M/M_0}(w_{M/K})}.$$

Therefore, for any increasing net $\underline{w}$ in $\mathbb{R}_{\geq 0}$ (all the nets that we consider in this work are indexed over the directed set $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, where $(n', d') \preceq (n, d)$ if $n' \leq n$ and $d' \mid d$), the system

$$\left\{\nabla_{\Gamma_d^{(n)}/K}^{\psi_{\Gamma_d^{(n)}/K_d^{nr}}(w_{(n,d)})} ; \mathscr{N}_{(n,d)/(n',d')}^{\text{Fesenko/Coleman}}\right\}_{\substack{n' \leq n \\ d' \mid d}} \tag{8.3}$$

is projective. Let

$$\begin{aligned}
\left(\nabla_K^{(\varphi)}\right)^{\underline{w}} &:= \nabla_K^{\underline{w}} \\
&= \varprojlim_{(n,d)} \nabla_{\Gamma_d^{(n)}/K}^{\psi_{\Gamma_d^{(n)}/K_d^{nr}}(w_{(n,d)})} \\
&= \widehat{\mathbb{Z}} \times \varprojlim_{(n,d)} \left(U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}\right)^{\psi_{\Gamma_d^{(n)}/K_d^{nr}}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{nr}}/Y_{\Gamma_d^{(n)}/K_d^{nr}}
\end{aligned} \tag{8.4}$$

be the projective limit of the system (8.3). Moreover, for (n, d), (n', d') satisfying $n' \leq n$, $d' \mid d$, and for $0 \leq w_{(n,d)}, w_{(n',d')} \in \mathbb{R}$ satisfying $w_{(n',d')} \leq w_{(n,d)}$ and $0 \leq \psi_{\Gamma_d^{(n)}/K}(w_{(n,d)})$, $\psi_{\Gamma_{d'}^{(n')}/K}(w_{(n',d')}) \in \mathbb{Z}$, the following square

$$\begin{array}{ccc}
\text{Gal}(\Gamma_d^{(n)}/K)^{w_{(n,d)}} & \xrightarrow[\sim]{\mathbf{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}} & A_{w_{(n,d)}} \\
\Big\downarrow {\scriptstyle \text{res}_{\Gamma_{d'}^{(n')}}} & & \Big\downarrow {\scriptstyle \mathscr{N}^{\text{Fesenko/Coleman}}_{(n,d)/(n',d')}} \\
\text{Gal}(\Gamma_{d'}^{(n')}/K)^{w_{(n',d')}} & \xrightarrow[\sim]{\mathbf{\Phi}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}} & A_{w_{(n',d')}}
\end{array} \tag{8.5}$$

is commutative, where for $(n, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ and for $0 \leq w_{(n,d)} \in \mathbb{R}$ satisfying $0 \leq \psi_{\Gamma_d^{(n)}/K}(w_{(n,d)}) \in \mathbb{Z}$,

$$A_{w_{(n,d)}} := \left\langle 1_{K^\times/N_{K_d^{nr}/K} K_d^{nr\times}} \right\rangle \times \left(U^{\diamond}_{\widetilde{\mathbb{X}}(\Gamma_d^{(n)}/K)}\right)^{\psi_{\Gamma_d^{(n)}/K}(w_{(n,d)})} Y_{\Gamma_d^{(n)}/K_d^{nr}}/Y_{\Gamma_d^{(n)}/K_d^{nr}}.$$

For any increasing net $\underline{w} = (w_{(n,d)})$ in $\mathbb{R}_{\geq 0}$, let ${}_1\nabla_K^{\underline{w}}$ denote the kernel of the projection $\mathrm{Pr}_1 : \nabla_K^{\underline{w}} \to \widehat{\mathbb{Z}}$. Thus, by the commutativity of the square (8.5), it follows that

$$\mathbf{\Phi}_K^{(\varphi)}\left(G_K^{\underline{w}}\right) = {}_1\nabla_K^{\underline{w}},$$

for any increasing net $\underline{w}$ in $\mathbb{R}_{\geq 0}$.

On the other hand, following closely [12], for every $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ and for each $0 \leq i \in \mathbb{Z}$, introduce the subgroup

$$\left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)_i := \left\{(0, \xi(X)) \in \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K} : \xi(X) \equiv X \pmod{X^{i+1}}\right\}$$

of $\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}$. Then, Laubie reciprocity map $\mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K} : \mathrm{Gal}(\Gamma_d^{(n)}/K) \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}$ for the extension $\Gamma_d^{(n)}/K$ restricts to an isomorphism

$$\mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K} : \mathrm{Gal}(\Gamma_d^{(n)}/K)^{\varphi_{\Gamma_d^{(n)}/K}(i)} = \mathrm{Gal}(\Gamma_d^{(n)}/K)_i \xrightarrow{\sim} \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)_i, \tag{8.6}$$

for every $0 \leq i \in \mathbb{Z}$. Therefore, by Proposition 8.1 and by (8.6), the comparison isomorphism $\mathscr{C}^{(\varphi)}_{n,d} = \mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K} \circ \left(\mathbf{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{-1} : \nabla^{(\varphi)}_{\Gamma_d^{(n)}/K} \xrightarrow{\sim} \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}$ with respect to the extension $\Gamma_d^{(n)}/K$ restricts, for each $0 \leq i \in \mathbb{Z}$, to an isomorphism

$$\mathscr{C}^{(\varphi)}_{n,d} : A_{\varphi_{\Gamma_d^{(n)}/K}(i)} \xrightarrow{\sim} \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)_i = \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i)}, \tag{8.7}$$

where

$$\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i) := \int_0^i \left(\left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)_0 : \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)_x\right)^{-1} dx$$

(for details look at Section 3.3 of [12]), and which clearly makes the following triangle

$$\begin{array}{ccc}
 & & A_{\varphi_{\Gamma_d^{(n)}/K}(i)} \\
 & \overset{\mathbf{\Phi}^{(\varphi)}_{\Gamma_d^{(n)}/K}}{\underset{\sim}{\nearrow}} & \\
\mathrm{Gal}(\Gamma_d^{(n)}/K)_i & & \Big\downarrow \mathscr{C}^{(\varphi)}_{n,d} \\
 & \overset{\sim}{\underset{\mathbf{\Lambda}^{(\varphi)}_{\Gamma_d^{(n)}/K}}{\searrow}} & \\
 & & \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i)}
\end{array}$$

commutative.

Now, by the commutativity of the diagram (1.5) and by (8.7), for every ordered-pair of positive integers (n,d) and (n',d') satisfying $n' \leq n$ and $d' \mid d$, the map $\mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')} : \mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K} \to \mathscr{G}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}$ restricts, for each $0 \leq i, i' \in \mathbb{Z}$ satisfying $i' \leq i$, to

$$\mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')} : \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i)} \to \left(\mathscr{G}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_{d'}^{(n')}/K}(i')}, \tag{8.8}$$

and the following square

$$\begin{array}{ccc} A_{\varphi_{\Gamma_d^{(n)}/K}(i)} & \xrightarrow[\sim]{\mathscr{C}_{n,d}^{(\varphi)}} & \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i)} \\ \Big\downarrow \mathscr{N}^{\mathrm{Fesenko/Coleman}}_{(n,d)/(n',d')} & & \Big\downarrow \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')} \\ A_{\varphi_{\Gamma_{d'}^{(n')}/K}(i')} & \xrightarrow[\sim]{\mathscr{C}_{n',d'}^{(\varphi)}} & \left(\mathscr{G}^{(\varphi)}_{\Gamma_{d'}^{(n')}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_{d'}^{(n')}/K}(i')} \end{array} \tag{8.9}$$

is commutative. Therefore, by (8.8), for an increasing net $\underline{i} = (i_{(n,d)})$ in $\mathbb{Z}_{\geq 0}$, the system

$$\left\{\left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i_{(n,d)})} ; \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n \\ d'|d}}$$

is projective, and by (8.9), the collection of topological isomorphisms

$$\left\{\mathscr{C}_{n,d}^{(\varphi)} : A_{\varphi_{\Gamma_d^{(n)}/K}(i_{(n,d)})} \xrightarrow{\sim} \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i)}\right\}_{(n,d)}$$

turns out to be an isomorphism

$$\left\{\mathscr{C}_{n,d}^{(\varphi)} : A_{\varphi_{\Gamma_d^{(n)}/K}(i_{(n,d)})} \xrightarrow{\sim} \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i)}\right\}_{(n,d)} :$$

$$\left\{A_{\varphi_{\Gamma_d^{(n)}/K}(i_{(n,d)})}; \mathscr{N}^{\mathrm{Fesenko/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n \\ d'|d}} \xrightarrow{\sim} \left\{\left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i_{(n,d)})} ; \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n \\ d'|d}}$$

between $\left\{A_{\varphi_{\Gamma_d^{(n)}/K}(i_{(n,d)})}; \mathscr{N}^{\mathrm{Fesenko/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n \\ d'|d}}$ and $\left\{\left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i_{(n,d)})} ; \mathscr{N}^{\mathrm{Laubie/Coleman}}_{(n,d)/(n',d')}\right\}_{\substack{n'\leq n \\ d'|d}}$.

Now, passing to the projective limits, the comparison isomorphism $\mathscr{C}_K^{(\varphi)} : \nabla_K^{(\varphi)} \xrightarrow{\sim} \mathscr{G}_K^{(\varphi)}$ with respect to K restricts to

$$\mathscr{C}_K^{(\varphi)} = \varprojlim_{(n,d)} \mathscr{C}_{n,d}^{(\varphi)} : {}_1\left(\nabla_K^{(\varphi)}\right)^{\underline{\varphi}(\underline{i})} = \varprojlim_{(n,d)} A_{\varphi_{\Gamma_d^{(n)}/K}(i_{(n,d)})} \xrightarrow{\sim} \left(\mathscr{G}_K^{(\varphi)}\right)^{\underline{\varphi}^{\mathrm{Laubie}}(\underline{i})} = \varprojlim_{(n,d)} \left(\mathscr{G}^{(\varphi)}_{\Gamma_d^{(n)}/K}\right)^{\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i_{(n,d)})},$$

where $\underline{\varphi}(\underline{i})$ and $\underline{\varphi}^{\mathrm{Laubie}}(\underline{i})$ denote the nets $\left(\varphi_{\Gamma_d^{(n)}/K}(i_{(n,d)})\right)$ and $\left(\varphi^{\mathrm{Laubie}}_{\Gamma_d^{(n)}/K}(i_{(n,d)})\right)$ respectively corresponding to the increasing net $\underline{i} = (i_{(n,d)})$ in $\mathbb{Z}_{\geq 0}$, and makes the following triangle

$$
\begin{array}{ccc}
 & & {}_1\left(\nabla_K^{(\varphi)}\right)^{\underline{\varphi}(\underline{i})} \\
 & \overset{\mathbf{\Phi}_K^{(\varphi)}}{\underset{\sim}{\nearrow}} & \\
G_K^{\underline{\varphi}(\underline{i})} & & \Big\downarrow \mathscr{C}_K^{(\varphi)} \\
 & \overset{\sim}{\underset{\mathbf{\Lambda}_K^{(\varphi)}}{\searrow}} & \\
 & & \left(\mathscr{G}_K^{(\varphi)}\right)^{\underline{\varphi}^{\mathrm{Laubie}}(\underline{i})}
\end{array}
$$

commutative. This completes the proof of Theorem 1.4.

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