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ON THE STANDARD L-FUNCTIONS OF CENTRAL-SIMPLE ALGEBRAS (*)

BY

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ABSTRACT. – In this article, we investigate (up to $\overline{\mathbb{Q}}$) the values of the standard L-function $L_{M_m(K)}(s,\Psi)$ of the algebra $M_m(K)$ over a *CM*-field K attached to a certain Hecke character Ψ of K (defined through a Hecke character ψ of the maximal totally real subfield F of K) at $s=(k+2n)/\eta(\mathfrak{k})$; where $m \leqslant k \in \mathbb{Z}$ is fixed and depends on the character ψ, $\forall n \in \mathbb{Z}^+$. © 2000 Éditions scientifiques et médicales Elsevier SAS

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We investigate special values of the standard L-function of a matrix algebra $M_m(K)$ over a *CM*-field K, and obtain a formula (Theorem 3.1), which generalizes a well-known result for Hecke L-functions (Corollary 3.3). However, we should point out that, our results are preliminary at best, since compared to automorphic L-functions, L-function of a matrix algebra is a relatively mild generalization of a classical Hecke L-function.

1. Arithmetic automorphic forms

We will be faithful to the notation of [6] and [8]. In particular, let F be a totally real number field and K a *CM*-field with maximal totally real subfield F. For simplicity, let $\mathfrak{k}$ denote F or K. Let **a** denote the set

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$\{\nu : F \hookrightarrow \mathbb{R}\}$ of archimedean primes of F, if $\mathfrak{k} = F$; or a fixed CM-type of K, if $\mathfrak{k} = K$. Consider the group $\widetilde{G} = \{g \in GL_{2m}(\mathfrak{k}) \colon\ g^* J_m g = J_m\}$ where

$$J_m = \begin{pmatrix} 0 & -1_m \\ 1_m & 0, \end{pmatrix}, \qquad g^* = {}^t g^\gamma$$

with $\mathrm{Gal}(\mathfrak{k}/F) = \langle\gamma\rangle$; and put $G = \widetilde{G} \cap SL_{2m}(\mathfrak{k})$. Let $\widetilde{P}$ be the parabolic subgroup of $\widetilde{G}$ corresponding to the partition (m, m) of $2m$, $\widetilde{Q}$ the Levi component and $\widetilde{R}$ the unipotent radical of $\widetilde{P}$. Likewise, define the subgroups P, Q and R of G. For the group G, we consider the Lie group $L = GL_m(\mathbb{C})^{[\mathfrak{k} : F]}$ and take a rational representation

$$\rho : L^{\mathbf{a}} = \overbrace{L \times \cdots \times L}^{\#(\mathbf{a})\ \text{copies}} \to GL(V),$$

where V is a finite-dimensional $\mathbb{C}$-space. Fix a $\overline{\mathbb{Q}}$-rational structure of V (let $V(\overline{\mathbb{Q}})$ denote the set of all $\overline{\mathbb{Q}}$-rational elements of V) such that ρ is a $\overline{\mathbb{Q}}$-rational representation with respect to the standard $\overline{\mathbb{Q}}$-rational structure of $L^{\mathbf{a}}$. Then we set (following [8]):

$$\text{(1a)} \qquad \Lambda_\rho(\alpha, Z) = \begin{cases} \rho((\mu_\nu(\alpha, Z))_{\nu\in\mathbf{a}}), & \text{if } \mathfrak{k} = F, \\ \rho((\kappa_\nu(\alpha, Z), \mu_\nu(\alpha, Z))_{\nu\in\mathbf{a}}), & \text{if } \mathfrak{k} = K, \end{cases}$$

for $\alpha \in G_{\mathbb{A}}$ (the adelization of G à lá Shimura) and

$$Z \in H_m^{\mathbf{a}} = \overbrace{H_m \times \cdots \times H_m}^{\#(\mathbf{a})\ \text{copies}},$$

where H_m denotes the symmetric space $H_m = H_m^{(\nu)} = \{x + iy \in S_\nu \otimes_{\mathbb{R}} \mathbb{C} \mid x, y \in S_\nu,\ \ y > 0\}$ $(\nu \in \mathbf{a})$ with $S = \{m \in M_m(\mathfrak{k}) \mid m^* = m\}$. Recall that, for $g \in G_\nu$ and $Z \in H_m^{(\nu)}$ $(\nu \in \mathbf{a})$, the equation

$$\begin{aligned} g\begin{pmatrix} Z^* & Z \\ 1_m & 1_m \end{pmatrix} &= \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}\begin{pmatrix} Z^* & Z \\ 1_m & 1_m \end{pmatrix} \\ &= \begin{pmatrix} g(Z)^* & g(Z) \\ 1_m & 1_m \end{pmatrix}\begin{pmatrix} \overline{\kappa_\nu(g, Z)} & 0 \\ 0 & \mu_\nu(g, Z) \end{pmatrix} \end{aligned} \qquad \text{(1b)}$$

defines the action of the archimedean groups G_ν on $H_m^{(\nu)}$ and the $GL_m(\mathbb{C})$-valued holomorphic factors of automorphy

$$\kappa_\nu, \mu_\nu : G_\nu \times H_m^{(\nu)} \to GL_m(\mathbb{C}).$$

Then define for $\alpha \in G$, the α-transform $f\|_\rho \alpha : H_m^{\mathbf{a}} \to V$ of a function $f : H_m^{\mathbf{a}} \to V$ as usual by $(f\|_\rho \alpha)(Z) = \Lambda_\rho(\alpha, Z)^{-1} f(\alpha(Z))$ for $Z \in H_m^{\mathbf{a}}$. Let $\mathfrak{U}_\rho(\Gamma)$ denote the space of all meromorphic functions $f : H_m^{\mathbf{a}} \to V$ which satisfy $(f\|_\rho \alpha)(Z) = f(Z)$, $\forall Z \in H_m^{\mathbf{a}}$ and $\forall \alpha \in \Gamma$ (here Γ is a congruence subgroup of G) and which are meromorphic at all cusps when $G \xrightarrow{\sim} SL_2(\mathbb{Q})$. Put $\mathfrak{U}_\rho = \bigcup_\Gamma \mathfrak{U}_\rho(\Gamma)$, where Γ runs over all congruence subgroups of G.

Let A be a finite-dimensional commutative semi-simple algebra over $\mathbb{Q}$. Let $A = \bigoplus_{1 \leqslant i \leqslant t} C_i$ where C_i is a number field. The set $\mathrm{Hom}(A, \mathbb{C})$ of algebra homomorphisms will be denoted by J_A; and we will denote the $\mathbb{Z}$-module $\mathbb{Z}[J_A]$ generated by the set J_A by I_A. Then there exists an embedding $I_A \hookrightarrow (A^\times)^\vee$ of I_A into the character group $(A^\times)^\vee$ of $A^\times$. Now assume that $A = \bigoplus_{1 \leqslant i \leqslant t} \mathfrak{k}_i$ where $\mathfrak{k}_i$ is a CM-field such that $\mathfrak{k}_i \supseteqq \mathfrak{k}$ and $\dim_{\mathfrak{k}}(A) = 2m$. Let $p_A : I_A \times I_A \to \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ be Shimura's period symbol ([8] and [10]). Consider the automorphism $p : A \to A$ making the square

(1c)
$$\begin{array}{ccc} A & \xrightarrow{\ p\ } & A \\ \uparrow{\scriptstyle \text{inc.}} & & \uparrow{\scriptstyle \text{inc.}} \\ \mathfrak{k}_i & \xrightarrow{\text{complex conj.}} & \mathfrak{k}_i \end{array}$$

commutative for every $1 \leqslant i \leqslant t$, which will be called as the "complex conjugation of A". Let $\delta : A \to A$ be a positive involution of A and $h : A \to M_{2m}(\mathfrak{k})$ be an injective $\mathfrak{k}$-linear homomorphism such that $h(x^\delta) = J_m h(x)^* J_m^{-1}$, $\forall x \in A$. Then, for

$$A[\delta] = \{a \in A \mid a.a^\delta = 1\},$$

the image $h(A[\delta]) \subset G$ and there exists a unique $w \in H_m^{\mathbf{a}}$ such that $g(w) = w$, $\forall g \in h(A[\delta])$; which is called a CM-point of G on $H_m^{\mathbf{a}}$. Let $(H_m^{\mathbf{a}})_{CM}$ denote the set of all CM-points of G on $H_m^{\mathbf{a}}$. In particular, for any injective homomorphism $h : A \to M_{2m}(\mathfrak{k})$ such that $h(p(x)) = J_m h(x)^* J_m^{-1}$, $\forall x \in A$; let $w \in (H_m^{\mathbf{a}})_{CM}$ be the fixed point of the action of $h(A[p])$ on $H_m^{\mathbf{a}}$. Then, there corresponds a $\overline{\mathbb{Q}}$-rational representation $A[p] \to GL(V)$ defined by $a \mapsto \Lambda_\rho(h(a), w)$ for $a \in A[p]$. Fixing a basis of V over $\mathbb{C}$, assume that $V \xrightarrow{\sim} \mathbb{C}^d$ and $V(\overline{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}}^d$. Then, there exists $U \in GL_n(\overline{\mathbb{Q}})$ and $\sigma_1, \ldots, \sigma_d \in I_A$ such that $\Lambda_\rho(h(a), w) = U \operatorname{diag}[a^{\sigma_1}, \ldots, a^{\sigma_d}]U^{-1}$ for every $a \in A[p]$. Choose a CM-type ∞ of A which is defined by $\infty = \infty_1 + \cdots + \infty_t$ where ∞_i is a CM-type of $\mathfrak{k}_i$ for $1 \leqslant i \leqslant t$ satisfying

$$\text{(1d)} \quad \prod_{1 \leqslant i \leqslant t} a^{\infty_i} = \begin{cases} \prod_{\nu \in \mathbf{a}} \det[\mu_\nu(h(a), w)], & \mathfrak{k} = F, \\ \prod_{\nu \in \mathbf{a}} \det[\kappa_\nu(h(a), w)] \det[\mu_\nu(h(a), w)], & \mathfrak{k} = K. \end{cases}$$

Then, letting

$$Q_\rho(w) = U \operatorname{diag}\big[p_A(\sigma_1, \infty), \ldots, p_A(\sigma_d, \infty)\big] U^{-1} \in GL_d(\mathbb{C}),$$

we call $f \in \mathfrak{U}_\rho(\Gamma)$ (where Γ is a congruence subgroup of G) arithmetic, if $Q_\rho(w)^{-1} f(w) \in \overline{\mathbb{Q}}^d$ for every $w \in (H_m^{\mathbf{a}})_{CM}$ where f is holomorphic. The space of all arithmetic elements $f \in \mathfrak{U}_\rho(\Gamma)$ is denoted by $\mathfrak{U}_\rho(\Gamma, \overline{\mathbb{Q}})$ and we put $\mathfrak{U}_\rho(\overline{\mathbb{Q}}) = \bigcup_\Gamma \mathfrak{U}_\rho(\Gamma, \overline{\mathbb{Q}})$ where Γ runs over all congruence subgroups of G.

2. Linear equivalence of Eisenstein series

We will consider the following special Langlands type Eisenstein series attached to G, defined with respect to the data $(k, \mathfrak{c}, \chi)$ consisting of an integral weight $k \in \mathbb{Z}^{\mathbf{a}}$, an integral ideal $\mathfrak{c}$ in F and a Hecke character $\chi : F_{\mathbb{A}}^\times \to \mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$ defined modulo $\mathfrak{c}$ satisfying the condition

$$\text{(2a)} \qquad \chi_{\mathbf{a}}(x) = \prod_{\nu \in \mathbf{a}} \left(\frac{x_\nu}{|x_\nu|}\right)^{k_\nu}, \quad \forall x \in F_{\mathbb{A}}^\times$$

which has the form:

$$\text{(2b)} \quad \begin{aligned} &E\big(Z, s; k, \chi, D^n[O_F, \mathfrak{c}]\big) \\ &\quad = \sum_{\zeta \in Z_{O_F}} N\big(\mathrm{il}_{O_F}(\zeta)\big)^{2s} \sum_{\alpha \in S_\zeta} \chi[\alpha]\, \delta(Z)^{s\mathbf{u} - \frac{k}{2}} J_{k, 2s\mathbf{u} - k}(\alpha, Z)^{-1}, \end{aligned}$$

where $Z \in H_m^{\mathbf{a}},\ s \in \mathbb{C},\ Z_{O_F}$ a system of representatives of the double-coset space

$$P\backslash\big(G \cap P_{\mathbb{A}} D[O_F, \mathfrak{c}]\big)/\Gamma[O_F, \mathfrak{c}];$$

and for $\zeta \in Z_{O_F}$,

$$S_\zeta = \big(\zeta\Gamma[O_F, \mathfrak{c}]\zeta^{-1} \cap P\big)\backslash\zeta\Gamma[O_F, \mathfrak{c}].$$

Here $D[O_F, \mathfrak{c}]$ denotes the "level $\mathfrak{c}$" open subgroup of the maximal compact subgroup

$$D[O_F, O_F] = \prod_{\nu\in\mathbf{h}} \big(G_\nu \cap GL_{2m}(O_{\mathfrak{k}_\nu})\big) \times \prod_{\nu\in\mathbf{a}} \big\{g \in G_\nu \mid g(\mathbf{i}) = \mathbf{i}\big\}^{1}$$

(where $\mathbf{i} = (i1_m, \ldots, i1_m) \in H_m^{\mathbf{a}}$) of $G_{\mathbb{A}}$, which is defined by

$$D[O_F, \mathfrak{c}] = \prod_{\nu\in\mathbf{a}\cup\mathbf{h}} D_\nu[O_F, \mathfrak{c}],$$

where

$$\text{(2c)}\ D_\nu[O_F, \mathfrak{c}] = \begin{cases} \{g \in G_\nu \cap GL_{2m}(O_{\mathfrak{k}_\nu}) \mid c_g \in \mathfrak{c}_\nu M_m(O_{\mathfrak{k}_\nu})\}, & \nu \in \mathbf{h}, \\ \{g \in G_\nu \mid g(\mathbf{i}) = \mathbf{i}\}, & \nu \in \mathbf{a}; \end{cases}$$

the ideal $\mathrm{il}_{O_F}(\zeta)$ in F is defined by $\mathrm{il}_{O_F}(\zeta) = \det(d_\pi)O_F \in I(F)$ where $\zeta = \pi\delta$ for some $\pi \in P_{\mathbb{A}}$ and $\delta \in D_o[O_F, O_F]$; and if $\alpha \in S_\zeta$ for $\zeta \in Z_{O_F}$, we put

$$\text{(2d)}\qquad \chi[\alpha] = \begin{cases} \chi_{\mathbf{a}}(\det(d_\alpha))\chi^*(\det(d_\alpha)\mathrm{il}(\zeta)^{-1}), & \mathfrak{c} \neq O_F, \\ \chi^*(\mathrm{il}(\zeta)^{-1}), & \mathfrak{c} = O_F, \end{cases}$$

[1] We remark that, by class field theory, a prime ideal $\mathfrak{p}$ in F splits completely in the extension $\mathfrak{k}/F$ if and only if $\mathfrak{p} \in \ker(A_{\mathfrak{k}/F})$, where $A_{\mathfrak{k}/F} : I_{d(\mathfrak{k}/F)}(F) \to \mathrm{Gal}(\mathfrak{k}/F)$ denotes the Artin reciprocity map ($d(\mathfrak{k}/F)$: the discriminant of the extension $\mathfrak{k}/F$). Thus, for $\nu \in \mathbf{h}$, $\mathfrak{k}_\nu \xrightarrow{\sim} F_\nu^{[\mathfrak{k}:F]}$ (as an F_ν-algebra) if and only if $\nu \in \ker(A_{\mathfrak{k}/F})$. So, for the case $\mathfrak{k} = K$; if $\nu \in \ker(A_{K/F})$, there is an F_ν-rational isomorphism $G_\nu \xrightarrow{\sim} \{x = (x_1, x_2) \in SL_{2m}(F_\nu)^2 \mid x_1 J_m x_2^* = J_m\}$ and if $\nu \notin \ker(A_{K/F})$, then there is an F_ν-rational isomorphism $G_\nu \xrightarrow{\sim} SU(m, K_\omega)$, where ω is a prime in K such that $\omega \mid \nu$ and K_ω corresponds to a quadratic extension of F_ν.

$\chi^*: I(F) \to \mathbb{T}$ being the ideal character corresponding to the Hecke character $\chi: F_{\mathbb{A}}^\times \to \mathbb{T}$. By Langlands' theory of Eisenstein series ([1] and [4]), the series $E(Z,s)$ converges for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to the whole complex plane as a function in s. The following proposition is easy to prove using the results in [8].

PROPOSITION 2.1. – *Assume that the integral weight $k \in \mathbb{Z}^{\mathbf{a}}$ satisfies $k_\nu = \kappa \in \mathbb{Z}$ ($\forall \nu \in \mathbf{a}$) such that*

$$\kappa \geqslant \begin{cases} \frac{m+1}{2}, & \text{if } \mathfrak{k} = F, \\ m, & \text{if } \mathfrak{k} = K, \end{cases}$$

and if $\kappa = (m+2)/2$ in the case $\mathfrak{k} = F$, we further suppose $F \neq \mathbb{Q}$ or $\chi^2 \neq 1$. Then, for $\omega \in (H_m^{\mathbf{a}})_{CM}$ and $f \in \mathfrak{U}_\kappa(\overline{\mathbb{Q}})$ such that $f(\omega) \neq 0$,

(1°) *$E(Z,s;k,\chi,D^m[O_F,\mathfrak{c}])$ is finite at $s = \kappa/2$;*

(2°) *$f(\omega)^{-1} E(\omega,\kappa/2;k,\chi,D^m[O_F,\mathfrak{c}]) \in \overline{\mathbb{Q}}$.*

In [2] and [3], we essentially studied the vector space $\mathcal{L}$ spanned by the G-transforms of the Langlands type Eisenstein series $E(Z,s;k,\psi, D^m[O_F,\mathfrak{c}])$ attached to the group G, and proved that there exists a data $\{k,\Delta,\lambda\}$ where $k \in \mathbb{Z}^{\mathbf{a}}$ is an integral weight; Δ a subgroup of $GL_m(\mathfrak{k})$ satisfying the condition $\underline{d}(P \cap \beta\Gamma\beta^{-1}) \subset \Delta \subset \ker(\omega)$ where $\Gamma \subset \tilde{G}$ is a congruence subgroup, $\{\beta\}$ is a system of representatives of $\tilde{P}\backslash\tilde{G}/\Gamma$ and $\omega: GL_m(\mathfrak{k}) \to \mathbb{C}^\times$ is a certain homomorphism ("determinant"); $\lambda: W \to \mathbb{C}$ is a locally constant function such that $\lambda(\Delta w \gamma) = \lambda(w)$ for every $\gamma \in \Gamma$ and $w \in W$; defining a series

$$\mathcal{E}(Z,s;k,\lambda) = \sum_{x \in \Delta\backslash W} \delta(Z)^{s\mathbf{u}-\frac{k}{2}} \lambda(x) j[x,Z]^{-k} \big| j[x,Z] \big|^{-2s\mathbf{u}+k}, \tag{2e}$$

where $Z \in H_m^{\mathbf{a}}$ and $s \in \mathbb{C}$ which spans the space $\mathcal{L}$.

More precisely, choose an integral ideal $\mathfrak{c}$ in F such that

$$\text{if } \omega \in U_{\mathfrak{k}} \text{ with } \omega \equiv 1 \big(\mathrm{mod}\ i_{\mathfrak{k}/F}(\mathfrak{c})\big), \text{ then } \omega \in F. \tag{2f}$$

Furthermore, suppose that $2 \notin \mathfrak{c}$. Such an integral ideal $\mathfrak{c}$ in F clearly exists. Let $\psi: F_{\mathbb{A}}^\times \to \mathbb{T}$ be a Hecke character defined modulo the integral ideal $\mathfrak{c}$ in F satisfying the condition (2a) for some $k \in \mathbb{Z}^{\mathbf{a}}$. Then, there exists a Hecke character $\Psi : \mathfrak{k}_{\mathbb{A}}^{\times} \to \mathbb{T}$ defined modulo $i_{\mathfrak{k}/F}(\mathfrak{c}) = \mathfrak{c}O_{\mathfrak{k}}$ and with the property

$$\Psi_{\mathbf{a}}(x) = \prod_{\nu \in \mathbf{a}} \left(\frac{x_\nu}{|x_\nu|} \right)^{k_\nu}, \quad \forall x \in \mathfrak{k}_{\mathbb{A}}^{\times} \tag{2g}$$

such that, the square

$$\begin{array}{ccc} I_{i_{\mathfrak{k}/F}(\mathfrak{c})}(\mathfrak{k}) & \xrightarrow{\Psi^*} & \mathbb{T} \\ \uparrow{\scriptstyle i_{\mathfrak{k}/F}} & & \uparrow{\scriptstyle \mathrm{id}} \\ I_{\mathfrak{c}}(F) & \xrightarrow{\psi^*} & \mathbb{T} \end{array} \tag{2h}$$

is commutative. The linear equivalence relation of Eisenstein series proved in [3] is now stated as follows:

THEOREM 2.2. – *Put*

$$\mathrm{Cl}_{i_{\mathfrak{k}/F}(\mathfrak{c})}(\mathfrak{k}) = \bigsqcup_{1 \leqslant i \leqslant h} [\mathfrak{a}_i] \quad \textit{and} \quad X_i \in I_r(\mathcal{O})$$

(*the set of integral right ideals of* $\mathcal{O} = M_m(O_{\mathfrak{k}})$) *such that*, $\mathcal{N}_{M_m(\mathfrak{k})/\mathfrak{k}}(X_i) = \mathfrak{a}_i$ *for* $1 \leqslant i \leqslant h$. *Then*:

$$L_{M_m(\mathfrak{k})}\left(\frac{2s}{\eta(\mathfrak{k})}, \Psi\right) E\left(Z, s; k, \psi, D^m[O_F, \mathfrak{c}]\right) = \sum_{1 \leqslant i \leqslant h} \Psi^*(\mathfrak{a}_i) N(\mathfrak{a}_i)^{\frac{2s}{\eta(\mathfrak{k})}} \mathcal{E}(Z, s; k, \lambda_{\Psi, X_i}),$$

where the standard L-function of $M_m(\mathfrak{k})$ *is defined as usual by*:

$$L_{M_m(\mathfrak{k})}(s, \Psi) = \sum_{[\mathfrak{a}] \in \mathrm{Cl}(\mathfrak{k})} \sum_{\substack{Y \in I_\ell(\mathcal{O}) \\ Y \sim \mathfrak{a}}} \frac{\Psi^*(\det(Y))}{N(\det(Y))^s}$$

which converges for $\mathrm{Re}(s) > 1$ *and does not depend on the choice of the maximal* $O_{\mathfrak{k}}$*-order* $\mathcal{O}$ *in* $M_m(\mathfrak{k})$.

Now assume that $\mathfrak{k} = K$ and $k \in \mathbb{Z}^{\mathbf{a}}$ satisfies $k_\nu = \kappa$ for every $\nu \in \mathbf{a}$ for some $\kappa \in \mathbb{Z}$. Then

PROPOSITION 2.3. – *$\exists\omega \in (H_m^{\mathbf{a}})_{CM}$ such that for $m \leqslant \kappa$:*

$$\mathcal{E}\left(\omega, \frac{\kappa}{2}; k, \lambda\right) \sim_{\overline{\mathbb{Q}}} \pi^{t(\kappa)}\mathcal{P}(\kappa) f(\omega),$$

where $f \in \mathfrak{U}_\kappa(\overline{\mathbb{Q}})$ such that $f(\omega) \neq 0$; $t(\kappa) = [F:\mathbb{Q}](m\kappa - \frac{(m-1)m}{2})$ and $\mathcal{P}(\kappa) = p_K(\sum_{\nu\in\mathbf{a}} \nu, \sum_{\nu\in\mathbf{a}} \nu)^{m\kappa}$.

3. Standard L-functions of $M_m(K)$

Using the results of Section 2, we can describe the values of the standard L-function $L_{M_m(K)}(s, \Psi)$ of $M_m(K)$ attached to the Hecke character $\Psi : K_{\mathbb{A}}^\times \to \mathbb{T}$ (defined above) at certain rational points on the real line, using values of arithmetic automorphic forms at *CM*-points and period symbol, up to $\overline{\mathbb{Q}}^\times$. More precisely, we have

THEOREM 3.1. – *Let $\mathfrak{c}$ be an integral ideal of F satisfying the condition* (2f) *and assume that $2 \notin \mathfrak{c}$. Let $\psi : F_{\mathbb{A}}^\times \to \mathbb{T}$ be a Hecke character defined modulo $\mathfrak{c}$ where the archimedean component satisfies* (2a) *and let $\Psi : K_{\mathbb{A}}^\times \to \mathbb{T}$ denote the Hecke character of K defined modulo $i_{K/F}(\mathfrak{c})$ satisfying* (2g) *and making the square* (2h) *commutative. Then, for $m \leqslant k$:*

$$L_{M_m(K)}\left(\frac{k}{2}, \Psi\right) \sim_{\overline{\mathbb{Q}}} \pi^{t(k)}\mathcal{P}(k)\left[\sum_{1\leqslant i\leqslant h} \Psi^*(\mathfrak{a}_i)N(\mathfrak{a}_i)^{\frac{k}{2}}\right],$$

where $t(k) = [F:\mathbb{Q}](mk - \frac{(m-1)m}{2})$ and $\mathcal{P}(k) = p_K(\sum_{\nu\in\mathbf{a}} \nu, \sum_{\nu\in\mathbf{a}} \nu)^{mk}$ as in Proposition 2.3, *and $\mathrm{Cl}_{i_{K/F}(\mathfrak{c})}(K) = \bigsqcup_{1\leqslant i\leqslant h}[\mathfrak{a}_i]$ as in Theorem* 2.2.

Thus as an immediate consequence of Theorem 3.1, we obtain

COROLLARY 3.2. – *Under the notation of Theorem* 3.1, *for every $n \in \mathbb{Z}$ such that $m \leqslant k + 2n$:*

$$L_{M_m(K)}\left(\frac{k+2n}{2}, \Psi\right) \sim_{\overline{\mathbb{Q}}} \pi^{t(k+2n)}\mathcal{P}(k+2n)\left[\sum_{1\leqslant i\leqslant h} \Psi^*(\mathfrak{a}_i)N(\mathfrak{a}_i)^{\frac{k+2n}{2}}\right].$$

Furthermore, specializing to the case $m = 1$, the standard L-function $L_{M_m(K)}(s, \Psi)$ reduces to the Hecke L-function $L_K(s, \Psi)$ of K attached to the character $\Psi : K_{\mathbb{A}}^\times \to \mathbb{T}$, and obtain the following well-known result:

COROLLARY 3.3. – *Under the notation of Theorem* 3.1*, for every* $n \in \mathbb{Z}$ *such that* $1 \leqslant k + 2n$:

$$L_K\left(\frac{k+2n}{2}, \Psi\right) \sim_{\overline{\mathbb{Q}}} \pi^{[F:\mathbb{Q}](k+2n)} p_K\left(\sum_{\nu\in\mathbf{a}} \nu, \sum_{\nu\in\mathbf{a}} \nu\right)^{(k+2n)} \times \left[\sum_{1\leqslant i\leqslant h} \Psi^*(\mathfrak{a}_i) N(\mathfrak{a}_i)^{\frac{k+2n}{2}}\right].$$

In particular, if the Hecke character $\psi : F_{\mathbb{A}}^\times \to \mathbb{T}$ *defined modulo* $\mathfrak{c}$ *satisfies*

$$\psi_{\mathbf{a}}(x) = \prod_{\nu\in\mathbf{a}} \left(\frac{x_\nu}{|x_\nu|}\right)^2 = 1$$

for every $x \in F_{\mathbb{A}}^\times$ *(i.e.* $k = 2$*), then*

$$L_K(1+n, \Psi) \sim_{\overline{\mathbb{Q}}} \pi^{[K:\mathbb{Q}](1+n)} p_K\left(\sum_{\nu\in\mathbf{a}} \nu, \sum_{\nu\in\mathbf{a}} \nu\right)^{2(1+n)} \times \left[\sum_{1\leqslant i\leqslant h} \Psi^*(\mathfrak{a}_i) N(\mathfrak{a}_i)^{1+n}\right].$$

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