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TWO LEMMAS ON FORMAL POWER SERIES*

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Abstract

Let L be a local field and $\tilde{L}$ the completion of the maximal unramified extension of L. In this short note, we prove

(I) the sequences

(1″) $0 \to \mathcal{O}_L[[X_1,\ldots,X_n]] \to \mathcal{O}_{\tilde{L}}[[X_1,\ldots,X_n]] \overset{\phi_L-1}{\to} \mathcal{O}_{\tilde{L}}[[X_1,\ldots,X_n]] \to 0$

and

(2″)

$1 \to \mathcal{O}_L[[X_1,\ldots,X_n]]^x \to \mathcal{O}_{\tilde{L}}[[X_1,\ldots,X_n]]^{\,x} \overset{\phi_L-1}{\to} \mathcal{O}_{\tilde{L}}[[X_1,\ldots,X_n]] \to 1$

are exact, where θ_L is the Frobenius automorphism over L applied on the coefficients of $\mathcal{O}_{\tilde{L}}[[X_1,\ldots,X_n]]$, and $\theta_L - 1$ respectively denotes the mapping $\alpha \mapsto \alpha^{\theta L} - \alpha$ for (1″) and $\epsilon \mapsto \frac{\epsilon^{\phi L}}{\epsilon}$ for (2″);

(II) $\mathcal{C}^\circ(\tilde{L},h)$ being the group the Coleman power series of degree 0, the sequence

$$1 \to \mathcal{C}^\circ(L,h) \to \mathcal{C}^\circ(\tilde{L},h)C^\circ(\tilde{L},h) \to 1$$

is exact, where $\mathcal{C}^\circ(L,h) = \mathcal{O}_L[[X]]^x \cap \mathcal{C}^\circ(\tilde{L},h)$

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1. Introduction

The following is well-known (see for example [3]): Let L be a local field, and $\tilde{L}$ te completion of the maximal unramified extension of L in a fixed algebraic closure Ω of L. Then

$$\text{(i)} \qquad 0 \to \mathcal{O}_L \to \mathcal{O}_{\tilde{L}} \overset{\phi_L - 1}{\to} \mathcal{O}_{\tilde{L}} \to 0$$

and

$$\text{(ii)} \qquad 1 \to \mathcal{O}_L^x \to \mathcal{O}_{\tilde{L}}^x \overset{\phi_L - 1}{\to} \mathcal{O}_{\tilde{L}}^x \to 1$$

are both exact sequences, where ϕ_L stands for the Frobenius automorphism over L, and $\phi_L - 1$ respectively denotes the mapping $\alpha \mapsto \alpha^{\phi_L} - \alpha$ for (i), and $\epsilon \mapsto \frac{\epsilon^{\phi_L}}{\epsilon}$ for (ii).

In this section we give a formal extension of the above result to the case of formal power series ring. This fact may be well-known, but we did not notice any literature explicitly referring to this fact. So we believe that it might be worthwhile to report it.

We start with a lemma:

Lemma 1. *Let R be a commutative ring with unity, and α an automorphism of R such that*

(a) the additive endomorphism $\alpha - 1$ of R defined by $x \mapsto x^\alpha - x (x \in R)$;

(b) the multiplicative endomorphism $\alpha - 1$ of R^x defined by $\epsilon \mapsto \frac{\epsilon^\alpha}{\epsilon} (\epsilon \in R^x)$ are both surjective. Then, $R[[X]]$ and $R[[X]]^x$ being the formal power series ring over R and its group of units respectively, the sequences

$$(1) \qquad 0 \to R_\alpha[[X]] \to R[[X]] \overset{\alpha - 1}{\to} R[[X]] \to 0$$

and

$$(2) \qquad 1 \to R_\alpha[[X]]^x \to R[[X]]^x \overset{\alpha - 1}{\to} R[[X]]^x \to 1$$

are both exact, where the automorphism α is applied only on the coefficients, and R_α denotes the fixed subring of α.

Hence, α-1 is surjective on $R[[X]]$ and on $R[[X]]^x$.

Proof. The proof of the first case is trivial. As for the second case, let $\sum_{v=0}^{\infty} a_v X^v$ (with $a_0 \in R^x$) be any element in $R[[X]]^x$. We are looking for an element $\sum_{\mu=0}^{\infty} b\mu X^\mu$ (with $b_0 \in R^x$) in $R[[X]]^x$ satisfying

$$\text{(A)} \quad \sum_{\mu=0}^{\infty} b_\mu^\alpha X^\mu = \left(\sum_{v=0}^{\infty} a_v X^v\right)\left(\sum_{\mu=0}^{\infty} b_\mu X^\mu\right) = \sum_{\lambda=0}^{\infty}\left(\sum_{\mu=0}^{\lambda} a_{\lambda-\mu} b_\mu\right) X^\lambda.$$

For $\lambda = 0$, we have $b_0^\alpha = a_0 b_0$ whose solution b_0 can be found in R^x by the assumption. Assume that b_μ (for $\mu = 0, 1, \ldots, n-1$) have already been found in R. Then, for $\lambda = n$, the equation (A) takes the form

$$\text{(B)} \quad b_n^\alpha = a_0 b_n + \sum_{\mu=0}^{n-1} a_{n-\mu} b_\mu = \frac{b_0^\alpha}{b_0} b_n + \sum_{\mu=0}^{n-1} a_{n-\mu} b_\mu,$$

where the last term is in R by the induction assumption. From (B), we obtain

$$\text{(C)} \quad \left(\frac{b_n}{b_0}\right)^\alpha = \frac{b_n}{b_0} + \frac{\sum_{\mu=0}^{n-1} a_{n-\mu} b_\mu}{b_0^\alpha}.$$

Since $b_0 \in R^x$, the last term above is in R. By (a), we can find $\frac{b_n}{b_0} \in R$ satisfying the equation (C). This completes the proof of the existence of $\sum_{\mu=0}^{\infty} b_\mu X^\mu \in R[[X]]^x$ satisfying (A). Now, the exactness of the second sequence follows easily.

Note that $R[[X_1, \ldots X_n]]$ is naturally identified with $R[[X_1, \ldots, X_{n-1}]][[X_n]]$ (see [4], pp. 206). Hence, as a corollary, we get: □

Corollary 1. *Under the same assumptions and notation as above, the sequences*

(1′) $0 \to R_\alpha[[X_1, \ldots, X_n]] \to R[[X_1, \ldots, X_n]] \overset{\alpha-1}{\to} R[[X_1, \ldots X_n]] \to 0$

and

(2′) $1 \to R_\alpha[[X_1, \ldots, X_n]]^x \to R[[X_1, \ldots, X_n]]^x \overset{\alpha-1}{\to} R[[X_1, \ldots X_n]]^x \to 1$

are both exact.

Hence $\alpha - 1$ is surjective on $R[[X_1, \ldots, X_n]]$ and on $R[[X_1, \ldots, X_n]]^x$.

Corollary 2. *Let L be a local field, and $\tilde{L}$ the completion of the maximal unramified extension of L. Then, $\mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]]$ and $\mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]]^x$ being the formal power series ring over the ring of integers $\mathcal{O}_{\tilde{L}}$ in $\tilde{L}$ and its group of units respectively, the sequences*

(1′′) $0 \to \mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]] \to \mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]] \overset{\phi_L - 1}{\to} \mathcal{O}_{\tilde{L}}[[X_1, \ldots X_n]] \to 0$

and

(2′) $1 \to \mathcal{O}_L[[X_1, \ldots, X_n]]^x \to \mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]]^x \overset{\phi_L - 1}{\to} \mathcal{O}_{\tilde{L}}[[X_1, \ldots X_n]]^x \to 1$

are both exact, where $\mathcal{O}_L[[X_1, \ldots, X_n]]$ *is the ring of formal power series with coefficients in* $\mathcal{O}_L$ *and* ϕ_L *is the Frobenius automorphism over* L *applied on the coefficients of* $\mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]]$.

Hence, $\phi_L - 1$ *is surjective on* $\mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]]$ *and on* $\mathcal{O}_{\tilde{L}}[[X_1, \ldots, X_n]]^x$.

Corollary 3. *Let* $\bar{k}$ *be the algebraic closure of a finite field* k. *Then the sequences*

(1′′′) $0 \to k[[X_1, \ldots, X_n]] \to \bar{k}[[X_1, \ldots, X_n]] \overset{\phi_K - 1}{\to} \bar{k}[[X_1, \ldots, X_n]] \to 1$

and

(2′′) $1 \to k[[X_1, \ldots, X_n]]^x \to \bar{k}[[X_1, \ldots, X_n]]^x \overset{\phi_K - 1}{\to} \bar{k}[[X_1, \ldots X_n]]^x \to 1$

are exact, where ϕ_k *stands for the Frobenius automorphism over* k, *i.e* $\alpha \mapsto \alpha^q$ *for all* $\alpha \in \bar{k}$ *with* $q = \mid k \mid$, *applied on the coefficients of* $\bar{k}[[X_1, \ldots, X_n]]$.

Hence, $\phi_k - 1$ *is surjective on* $\bar{k}[[X_1, \ldots, X_n]]$ *and on* $\bar{k}[[X_1, \ldots, X_n]]^x$.

If the set of indeterminates is 0, then Corollary 2 reduces to the classical case mentioned in the beginning of the note.

2. Let L *be a local field, and let* $\tilde{L}$ *be the completion of the maximal unramified extension of* L *in a fixed algebraic closure* Ω *of* L. *The completion of* Ω *will be denoted by* $\hat{\Omega}$. *Further, let* π *be a prime of* $\mathcal{O}_{\tilde{L}}$, *the rign of integers in* $\tilde{L}, h$ *a Lubin-Tate series in* $\mathcal{O}_{\tilde{L}}[[X]]$, *and* $F_h(X, Y)$ *the (unique) formal group law belonging to* h. *Coleman in [1] introduced a norm operator* $\mathcal{N}_h : \mathcal{O}_{\tilde{L}}((X))^x \to \mathcal{O}_{\tilde{L}}((X))^x$ *by*

(iii) $(\mathcal{N}_h g) \circ h = \prod_{\omega \in W_h^1} g(X[+_h]\omega)$,

where $O_{\tilde{L}}((X))^x$ *is the multiplicative group of units in the ring of Laurent series over* $O_{\tilde{L}}, W_h^1 := \{\omega \in \hat{\Omega} : h(\omega) = 0\}$, *and* $X[+_h]\omega = F_h(X, \omega)$ *(see [2]).*

Then, $\mathcal{N}_h$ *is an endomorphism of* $\mathcal{O}_{\tilde{L}}((X))^x$ *satisfying* $\mathcal{N}_h g \equiv g^{\phi_L} (mod \pi)$ *for all* $g \in \mathcal{O}_{\tilde{L}}((X))^x$ *(see Theorem 11 in [1]). The Laurent series* $g \in \mathcal{O}_{\tilde{L}}((X))^x$ *satisfying* $\mathcal{N}_h g = g^{\phi_L}$ *are called Coleman power series, and the subgroup of* $\mathcal{O}_{\tilde{L}}((X))^x$ *consisting of all Coleman series is denoted by* $\mathcal{C}(\tilde{L}, h)$. *In particular,* $\mathcal{C}^\circ(\tilde{L}, h) := \mathcal{C}(\tilde{L}, h) \cap \mathcal{O}_{\tilde{L}}[[X]]^x$, *the group of Coleman series of degree 0, plays a fundamental role in the construction of the non-abelian local class field theory. The most significant feature of* $C(\tilde{L}, h)$ *is the* *following: upon reduction mod* $\pi, C(\tilde{L},h)$ *is isomorphically mapped onto* $\bar{\mathcal{F}}_q((X))^x$ *where* $\mathcal{F}_q$ *denotes the residue class field of* L*, and* $\bar{\mathcal{F}}_q$ *for its algebraic closure. Hence, upon reduction mod* $\pi, \mathcal{C}^\circ(\tilde{L},h)$ *is isomorphically mapped onto* $\bar{\mathcal{F}}_q[[X]]^x$*.*

Lemma 2. *Keeping all notations as above, assume that the Lubin-Tate formal power series* $h \in \mathcal{O}_L[[X]]$*. Then, the sequence*

$$1 \to \mathcal{C}^\circ(L,f) \to \mathcal{C}^\circ(\tilde{L},f) \stackrel{\phi_L-1}{\to} \mathcal{C}^\circ(\tilde{L},f) \to 1$$

is exact, where $\mathcal{C}^\circ(L,f) = \mathcal{O}_L[[X]]^x \cap \mathcal{C}^\circ(\tilde{L},f)$*.*

Proof. *It suffices to show the surjectivity of* $\phi_L - 1$*. Before doing this, some remarks will be in place. By the assumption, the Lubin-Tate series* h *is in* $\mathcal{O}_L[[X]]$*, i.e.* $h^{\phi_L} = h$*. Hence, by the remark on pp. 49 of [3],* $F_h(X,Y) \in \mathcal{O}_L[[X,Y]]$*. Furthermore,* $h^{\phi_L} = h$ *implies that* $\omega^{\phi_L} \in W_h^1$ *for all* $\omega \in W_h^1$*. Hence, for any* $\omega \in W_h^1$*,*

$$(X[+_h]\omega)^{\phi_L} = (F_h(X,\omega))^{\phi_L} = F_h^{\phi_L}(X,\omega^{\phi_L}) = F_h(X,\omega^{\phi_L}) = X[+_h]\omega^{\phi_L},$$

where $\omega^{\phi_L} \in W_h^1$*.*

From these remarks, it follows that $g^{\phi_L} \in \mathcal{C}^\circ(\tilde{L},h)$ *for every* $g \in \mathcal{C}^\circ(\tilde{L},h)$*. In fact, by (iii), we have*

$$\begin{aligned}
\mathcal{N}_h(g^{\phi_L}) \circ h &= \prod_{\omega \in W_h^1} g^{\phi_L}(X[+_h]\omega) \\
&= \prod_{\omega \in W_h^1} g^{\phi_L}((X[+_h]\omega)^{\phi_L} \\
&= (\prod_{\omega \in W_h^1} g(X[+_h]\omega))^{\phi_L} \\
&= (\mathcal{N}_h(g) \circ h)^{\phi_L} \\
&= (g^{\phi_L} \circ h)^{\phi_L} = (g^{\phi_L})^{\phi_L} \circ h),
\end{aligned}$$

so that $\mathcal{N}_h(g^{\phi_L}) - (g^{\phi_L})^{\phi_L}$ *(otherwise, the power series* $\mathcal{N}_h(g)^{\phi_L}) - (g^{\phi_L})^{\phi_L}$ *will have infinitely many zeroes with increasing exponential values). Hence* $g^{\phi_L} \in \mathcal{C}^\circ(\tilde{L},h)$ *as* *desired. Note that at the second equality before the last one, the assumption* $g \in \mathcal{C}^\circ(\tilde{L}, h)$ *was used. Now, returning to the proof of our assertion, take any* $f \in \mathcal{C}^\circ(\tilde{L}, h)$ *and set* f *(mod* π*)=*$\xi \in \bar{\mathcal{F}}_q[[X]]^x$. *Then, by Corollary 3 above, there is a* $\xi \in \bar{\mathcal{F}}_q[[X]]^x$ *satisfying* $\frac{\xi_1^{\phi_{F_q}}}{\xi_1} = \xi$. *Let* f_1 *be the unique element of* $\mathcal{C}^\circ(\tilde{L}, h)$ *such that* f_1 *(mod* π*)=*ξ_1. *Then* $f_1^{\phi_L} \in \mathcal{C}^\circ(\tilde{L}, h)$ *by the remark above, and* $\frac{f_1^{\phi_L}}{f_1} \in \mathcal{C}^\circ(\tilde{L}, h)$. *Furthermore, upon reduction mod* π, $\frac{f_1^{\phi_L}}{f_1}$ *yields* $\frac{\xi_1^{\phi_{F_q}}}{\xi_1} = \xi$, *so that* $\frac{f_1^{\phi_L}}{f_1} = f$ *by the uniqueness of the lifting. This proves our assertion.*

References

[1] R. Coleman, *Division values in local fields*, Invent. Math **53** (1979), 91-116.

[2] E. de Shalit, *Iwasawa Theory of Elliptic Curves with Complex Multiplication*, Perspectives in Math., vol. 3, Academic Press, Boston-Orlando, 1987.

[3] K. Iwasawa, *Local Class Field Theory,* Oxford University Press, Oxford-New York, 1986.

[4] S. Lang, *Algebra, 3rd* ed., Addison-Wesley, Reading-Massachusetts, 1995.

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