Turk J Math
24 (2000) , 25 – 58.

On the Metabelian Local Artin Map I: Galois Conjugation Law

*Kazım İlhan İkeda**

Abstract

It is proved that, for a (henselian) local field K and for a fixed Lubin-Tate splitting ϕ over K, the metabelian local Artin map $(?,\, K)_\phi : \mathfrak{G}(K,\phi) \tilde{\to} Gal(K^{(ab)^2}/K)$ satisfies the Galois conjugation law

$$(\tilde{\sigma}^{+}(\alpha), \sigma(K))_{\tilde{\sigma}\phi\tilde{\sigma}^{-1}} = \tilde{\sigma}|_{K^{(ab)2}}(\alpha, K)_\phi \tilde{\sigma}^{-1}|_{\tilde{\sigma}(K^{(ab)^2})}$$

for any $\alpha \in \mathfrak{G}(K,\phi)$, and for any embedding $\sigma : K \hookrightarrow K^{sep}$, where $\tilde{\sigma} \in \mathrm{Aut}\ (K^{sep})$ is a fixed extension to K^{sep} of the embedding $\sigma : K \hookrightarrow K^{sep}$.

Key words and phrases. local fields, metabelian extensions, metabelian local Artin map, non-abelian local class field theory.

§1. Introduction

Recall that a Galois extension F/K is called a metabelian (or 2-abelian) extension, if the double-commutator group $(\mathrm{Gal}(F/K)')' = \mathrm{Gal}(F/K)'^{(2)}$ is trivial. More generally, F/K is called an n-abelian extension if the n-th commutator group $\mathrm{Gal}(F/K)'^{(n)}$ is trivial. Recently, H. Koch and E. de Shalit constructed a class field theory for 2-abelian extensions of local fields [4]. The remarkable fact is that it seems to be possible to recover non-abelian local class field theory by inductively extending the results in [4] to n-abelian extensions of local fields for $2 < n \in \mathbb{Z}$. Let K be a (non-archimedean) local field with finite residue class field $\kappa_K = O_K/\mathfrak{p}_K$ of q_K elements. Here as usual, O_K and $\mathfrak{p}_K$ denote the ring of integers of K and the maximal ideal in O_K, respectively. Let K^{nr} denote

1991 *Mathematics Subject Classification.* 11S37.

*This work had been carried out and completed in the frame of the project "$GL(n)$" at Tübitak-Feza Gürsey Institute, İstanbul-Turkey.

the maximal unramified extension of K in a fixed seperable closure K^{sep} of $K, \tilde{K}$ the completion $\overline{K^{nr}}$ of K^{nr}, and let $K^{(ab)^2}$ denote the maximal 2-abelian extension of K (in K^{sep}). Following the terminology of [4], fix a Lubin-Tate splitting (K, ϕ) once and for all. Let K_ϕ be the fixed field of ϕ in K^{sep}. Recall that Lubin-Tate theory states that $K^{ab} = K^{nr}K(W)$, where $K(W)$ is a maximal totally ramified extension of K defined by adjoining the set of torsion points W defined by a Lubin-Tate polynomial $f_K \in O_K[[X]]$, and by the compositions $\phi^n f_K \circ \cdots \circ \phi f_K \circ f_K$ for $0 \leq n \in \mathbb{Z}$. The idea of Koch and de Shalit in [4] is based on the following observation: if L runs through the finite extensions of K contained in $K_\infty = K_\phi \cap K^{ab}$, then $\bigcup_{L,d}(LK_d^{nr})^{ab} = K^{(ab)^2}$ (for a generalization, cf. [1]), so that passing to the projective limits (defined over the canonical connecting morphisms)

$$\varprojlim_{L,d} \text{ Gal } ((LK_d^{nr})^{ab}/K) = \varprojlim_{d} \text{ Gal } ((K_\infty K_d^{nr})^{ab}/K) = \text{ Gal } (K^{(ab)^2}/K).$$

Hence if it can be found, (for each $1 \leq d \in \mathbb{Z}$) a compact group $G_d(K)$ topologically isomorphic to $\text{Gal}((K_\infty K_d^{nr})^{ab}/K)$ together with canonical topological isomorphisms

$$\iota_d : G_d(K) \xrightarrow{\sim} \text{ Gal } ((K_\infty K_d^{nr})^{ab}/K)$$

satisfying the functoriality condition, then $K \mapsto G(K) = \varprojlim_{d} G_d(K)$ will be the "non-abelian" field formation of a class field theory for 2-abelian extensions of the local field K. Indeed, Koch and de Shalit have defined such topological groups $G_d(K)$ (depending on the choice of the Lubin-Tate splitting (K, ϕ)) in the form of *d-Koch-de Shalit vectors* $\mathfrak{G}_d(K, \phi)$, and succeeded in building the metabelian (2-abelian) local class field theory via constructing a canonical isomorphism

$$\iota_{\phi,d} : \mathfrak{G}_d(K, \phi) \xrightarrow{\sim} \text{ Gal } ((K_\infty K_d^{nr})^{ab}/K),$$

and consequently establishing the 2-abelian local Artin map

$$(?.K)_\phi = \iota_\phi : \mathfrak{G}(K, \phi) = \varprojlim_{d} \mathfrak{G}_d(K, \phi) \xrightarrow{\sim} \text{ Gal } (K^{(ab)^2}/K). \tag{1.1}$$

However in [4], Koch and de Shalit neither discussed the behaviour of the morphism (1.1) under Galois conjugation, nor discussed the "metabelian transfer law" (Verlagerung) of the morphism (1.1). The aim of this work, and its continuation, is to discuss these two properties of the 2-abelian local Artin map, and to complete [4] in that respect. Thus in the first part, we will review the metabelian local class field theory following [4]. This part is included, since [4] is a very important and technical theory and will form the basis for our future investigation on non-abelian local class field theory. Then in the second part, after some preliminary observations on the group of Koch-de Shalit vectors $\mathfrak{G}(K,\phi)$, we will state and prove the Galois conjugation law of the metabelian local Artin map. The third part is devoted to the "metabelian transfer law" of the 2-abelian local Artin map. As we intend to include the necessary Galois cohomological tools in part *III,* we decided to split up our paper into two, and part *III* will appear elsewhere as the natural continuation of this paper which consists of part *I* and *II.*

Part I: Metabelian Local Class Field Theory

§2. Koch-de Shalit Vectors

Following [4], let L be a local field with finite residue class field κ_L of q_L elements, $\phi_L \in \mathrm{Gal}(L^{nr}/L)$ the Frobenius automorphism over L, and π a prime element of $\tilde{L}$ (the completion $\overline{L^{nr}}$ of L^{nr}).

Definition 2.1. A formal power series $h(X) \in O_{\tilde{L}}[[X]]$ is called a Lubin-Tate power series belonging to π for $\tilde{L}$, if

$$h(X) = \pi X + \text{ (higher-degree terms)}$$

and

$$h(X) \equiv X^{q_L} (\text{mod } \pi).$$

Let $\mathcal{F}'_\pi$ denote the collection of all Lubin-Tate formal power series belonging to π for $\tilde{L}$. If $h(X) \in \mathcal{F}'_\pi$, then there exists a unique 1-dimensional formal group law* $F_h(X,Y) \in O_{\tilde{L}}[[X,Y]]$ satisfying

*Let R be a commutative ring with unity 1_R. Recall that $F(X,Y) \in R[[X,Y]]$ is called a (commutative 1-dimensional) formal group law over R if the following axioms are satisfied: (a) $F(X,Y) = F(Y,X)$; (b) $F(X,Y) \equiv X+Y$ (mod deg. 2); (c) $F(X,0) = X = F(0,X)$; (d) $F(F(X,Y),Z) = F(X,F(Y,Z))$. Moreover, if R is a complete d.v.r and $\mathfrak{p}$ the maximal ideal of R, then for $\alpha,\beta \in \mathfrak{p}$, define $\alpha [\underset{F}{+}] \beta = F(\alpha,\beta) \in \mathfrak{p}$, which defines a new abelian group structure on $\mathfrak{p}$, called the group of $\mathfrak{p}$-valued points of

$$(\phi_L F_h)(h(X), h(Y)) = h(F_h(X, Y)),$$

which is called the Lubin-Tate formal group law over $O_{\tilde{L}}$ attached to the Lubin-Tate series $h(X)$. The endomorphism ring (over $O_{\tilde{L}}$)

$$\text{End}_{O_{\tilde{L}}}(F_h(X, Y)) = \{f(X) \in XO_{\tilde{L}}[[X]] : f(F_h(X, Y)) = F_h(f(X), f(Y))\}$$

is isomorphic to the ring of integers O_L of the local field L under the isomorphism $a \mapsto [a]_h$ for every $a \in O_L$, where $[a]_h : F_h \to F_h$ is the unique endomorphism of the form $[a]_h = aX+$ (higher-degree terms) $\in XO_{\tilde{L}}[[X]]$. Now, suppose that π' is another prime element of $\tilde{L}$, and $h'(X) \in \mathcal{F}'_{\pi'}$. If $0 \neq a \in O_{\tilde{L}}$ satisfies $\frac{\phi_L(a)}{a} = \frac{\pi'}{\pi}$, then there exists a unique homomorphism $[a]_{h,h'} : F_h \to F_{h'}$ of the form $[a]_{h,h'} = aX+$ (higher-degree terms) $\in XO_{\tilde{L}}[[X]]$ such that $(\phi[a]_{h,h'}) \circ h = h' \circ [a]_{h,h'}$ and $[ab]_{h,h''} = [b]_{h',h''} \circ [a]_{h,h'}$, where π'' is a prime element in $\tilde{L}$, possibly different than the prime elements π and π' of $\tilde{L}$, $h'' \in \mathcal{F}'_{\pi''}$, and $0 \neq b \in O_{\tilde{L}}$ chosen to satisfy $\frac{\phi_L(b)}{b} = \frac{\pi''}{\pi'}$.

Definition 2.2. For $1 \leq i \in \mathbb{Z}$, $x \in \mathfrak{p}_{\tilde{L}^{sep}}$ is called a torsion point of level i on F_h, if $\phi_L^{i-1} h \circ \cdots \circ \phi_L h \circ h(x) = 0$. The set of all torsion points of level i on F_h is denoted by W_h^i, and any $x \in W_h^i - W_h^{i-1} = \widetilde{W_h^i}$ is called a primitive torsion point of level i on F_h.

Observe that,

$$W_h^i = \{x \in \mathfrak{p}_{\tilde{L}^{sep}} : [a]_h(x) = 0, ^{\forall} a \in \mathfrak{p}_L^i\}.$$

Let L^i denote the abelian extension over L which is class field to the subgroup $U^i(L)$ of $L^\times$. Clearly, $L^{nr} \subseteq L^i$, since L^{nr} is the abelian extension over L which is class field to $U(L)$, and passing to completion, $\tilde{L} \subseteq \overline{L^i} = \widetilde{L^i}$ since $(L^i)^{nr} = L^i$. Lubin-Tate theory states that $\overline{L^i} = \tilde{L}(\omega)$ for any $\omega \in \widetilde{W_h^i}$ (where $h \in \mathcal{F}'_\pi$ is chosen arbitrarily for any prime element π of $\tilde{L}$). In the special case $\pi \in O_L$ and $h(X) \in \mathcal{F}'_\pi \cap O_L[[X]]$, $L_i = L(\omega)$ for any

$F(X, Y)$. Given formal group laws $F(X, Y)$ and $G(X, Y)$ over R, a formal power series $f(X) \in XR[[X]]$ is called a homomorphism from $F(X, Y)$ to $G(X, Y)$ (over R), and denoted by $f : F(X, Y) \to G(X, Y)$, if $f(F(X, Y)) = G(f(X), f(Y))$. The set $\text{Hom}_R(F, G)$ of all homomorphisms $F \to G$ over R is a group under the law of composition $(f, g) \mapsto f[\underset{G}{+}]g \in XR[[X]]$ for every $f, g \in \text{Hom}_R(F, G)$, and in the special case $F = G$, $\text{Hom}_R(F, F) = \text{End}_R(F)$ is a ring under the addition $[\underset{F}{+}]$ defined previously and composition as the multiplication.

$\omega \in \widetilde{W}_h^i$ is the abelian extension over L which is class field to the subgroup $< \pi > U^i(L)$ of $L^\times$, which is a totally-ramified extension over L of degree $[L_i : L] = (q_L - 1)q_L^{i-1}$, and $L^i = L^{nr}L_i = L^{nr}(\omega)$. For $h(X) \in \mathcal{F}'_\pi$, let

$$\Omega_h = \{\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, ...) : \omega_i \in \widetilde{W}^i_{\phi_L^{1-i}h} \text{ and } (\phi_L^{1-i}h)(\omega_i) = \omega_{i-1}\}.$$

The main result of Lubin-Tate theory states that

$$(u, L)^{-1}(\omega_i) = [u]_{\phi^{1-i}h}(\omega_i) \tag{2.1}$$

for every $u \in U(L)$ and for every $(\omega_i)_{1 \leq i \in \mathbb{Z}} \in \Omega_h$. A formal power series $h \in \mathcal{F}'_\pi$ is called a normic Lubin-Tate formal power series, if $\boldsymbol{\omega} \in \Omega_h$ are norm-compatible sequences.

Following the terminology introduced in [4], fix a Lubin-Tate splitting (K, ϕ) (that is a fixed extension of the Frobenius $\phi_K \in \text{Gal}(K^{nr}/K)$ over K to a K-automorphism ϕ of K^{sep}) and let K_ϕ be the fixed field of ϕ in K^{sep}(K_ϕ is totally-ramified over K). Then, the following facts (proved in [4]) listed below are crucial in what follows:

Lemma 2.3 (Koch-de Shalit).
(a) There exists a unique norm-compatible sequence of primes

$$\wp_\phi = \{\pi_L \in L^\times : K \subseteq L \subseteq K_\phi; [L : K] < \infty\};$$

that is, the Lubin-Tate splitting (K, ϕ) canonically determines a Lubin-Tate labelling $\overline{\wp}_\phi = \{\pi_L \in \tilde{L}^\times : K \subseteq L \subset K^{sep}; [L : K] < \infty\}$;
(b) For any $L \subseteq K_\phi, [L : K] < \infty$, there exists a unique Lubin-Tate formal power series $f_{\phi,L} \in O_L[[X]]$ belonging to π_L (chosen as in (a)) for L, such that $f_{\phi,L}(\pi_{L_n}) = \pi_{L_{n-1}}$ for every $n \geq 2$, $f_{\phi,L}(\pi_{L_1}) = 0$. Here, L_n denotes the class-field to the subgroup $< \pi_L > U^n(L)$ of $L^\times$, which is a totally-ramified abelian extension over L of degree $[L_n : L] = (q-1)q^{n-1}$, and explicitly given by $L_n = L(\omega)$ for any $\omega \in \tilde{W}^n_{f_{\phi,L}}$ (actually $L_n = L(\omega)$ for any $\omega \in \tilde{W}^n_f$ and for any $f \in \mathcal{F}'_{\pi_L} \cap O_L[[X]]$, where $\mathcal{F}'_{\pi_L}$ is the set of all Lubin-Tate formal power series belonging to π_L for $\tilde{L}$ and $\tilde{W}^n_f$ is the set of all primitive torsion points of level n in the unique Lubin-Tate formal group law F_f associated to $f \in \mathcal{F}'_{\pi_L} \cap O_L[[X]]$, which is defined over O_L and satisfies $F_f \circ (f \times f) = f \circ F_f$);
(c) in particular, there exists a unique Lubin-Tate formal group law $F_{f_{\phi,L}} = F_{\phi,L}$ defined over O_L associated to the unique Lubin-Tate formal power series $f_{\phi,L} \in O_L[[X]]$ satisfying $F_{\phi,L} \circ (f_{\phi,L} \times f_{\phi,L}) = f_{\phi,L} \circ F_{\phi,L}$ for each $K \subseteq L \subseteq K_\phi$ with $[L : K] < \infty$;

(d) for $a \in O_L$, there exists a unique endomorphism $[a]_{f_{\phi,L}} : F_{\phi,L} \to F_{\phi,L}$ over O_L of the Lubin-Tate formal group law $F_{\phi,L}$ of the form $[a]_{f_{\phi,L}} = aX+$ (higher degree terms) $\in XO_L[[X]]$ for each $K \subseteq L \subseteq K_\phi$ with $[L:K] < \infty$. Let $\{a\}_{f_{\phi,L}} = [a]_{f_{\phi,L}}$ (mod $\mathfrak{p}_L) \in X\kappa_L[[X]]$ be the reduction modulo $\mathfrak{p}_L$ of $[a]_{f_{\phi,L}} \in XO_L[[X]]$;
(e) every $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, ...) \in \Omega_{f_{\phi_K}}$ is norm-compatible.

For each $1 \le d \in \mathbb{Z}$, define a topological group structure on *d-Koch-de Shalit vectors* (with respect to (K,ϕ))

$$\mathfrak{G}_d(K,\phi) = \left\{ (a,\xi) : \begin{array}{l} a = u\pi_K^\nu \in \widehat{K^\times} \text{ s.t } u \in U(K), \nu \in \widehat{\mathbb{Z}} \\ \xi = \xi(X) \in \overline{\kappa}_K[[X]]^\times \text{ s.t.} \frac{\phi^d \xi}{\xi} = \frac{\{u\}_{f_{\phi,K}}}{X} \end{array} \right\}$$

(where $\widehat{K^\times}$ is the profinite completion of the multiplicative group $K^\times$) by the law of composition defined as

$$(a_1,\xi_1)(a_2,\xi_2) = (a_1.a_2, \xi_1.\phi^{-\nu_1}(\xi_2) \circ \{u_1\}_{f_{\phi,K}})$$

where $a_1 = u_1\pi_K^{\nu_1}, a_2 = u_2\pi_K^{\nu_2} \in \widehat{K^\times}$ with $u_1, u_2 \in U(K)$ and $\nu_1, \nu_2 \in \widehat{\mathbb{Z}}$ (=the profinite completion of $\mathbb{Z}$); and in which a basis of neighborhoods of the identity element of $\mathfrak{G}_d(K,\phi)$ is given by

$$\mathfrak{G}_d(K,\phi)^{(i,j)} = \{(a,\xi) \in \mathfrak{G}_d(K,\phi) : a \in U^i(K), \xi \equiv 1 (\text{mod } X^j)\}$$

for $0 \le i,j \in \mathbb{Z}$. By the results of paper [2], $\mathfrak{G}_d(K,\phi)$ is a non-void set. Note that, the identity element of $\mathfrak{G}_d(K,\phi)$ is $(1_K, 1(X))$, and the inverse of $(a,\xi) \in \mathfrak{G}_d(K,\phi)$ is $(a,\xi)^{-1} = (a^{-1}, \frac{1}{\phi^\nu(\xi)\circ\{u\}_{f_{\phi,K}}})$. For two positive integers d_1 and d_2 such that $d_1|d_2$, there exists transition morphism $\tau(\phi)_{d_1}^{d_2} : \mathfrak{G}_{d_2}(K,\phi) \to \mathfrak{G}_{d_1}(K,\phi)$ defined by

$$\tau(\phi)_{d_1}^{d_2} : (a,\xi) \mapsto \left(a, \prod_{0 \le i \le \frac{d_2}{d_1}-1} \phi^{d_1 i}(\xi) \right) \tag{2.2}$$

for every $(a,\xi) \in \mathfrak{G}_{d_2}(K,\phi)$. Let $\mathfrak{G}(K,\phi)$ denote the projective limit $\varprojlim_d \mathfrak{G}_d(K,\phi)$ defined over the transition morphisms eq. no. (2.2). In this paper, $\mathfrak{G}(K,\phi)$ will be called the group of *Koch-de Shalit vectors* (with respect to (K,ϕ)).

Now, suppose that L is a local field with finite residue class field κ_L of q_L elements, $\phi_L \in \mathrm{Gal}\,(L^{nr}/L)$ the Frobenius automorphism over L and π a prime element of $\tilde{L} = \overline{L^{nr}}$. For $h(X) \in \mathcal{F}'_\pi$, there exists a unique mapping

$$\mathcal{N}_h : O_{\tilde{L}}((X))^\times \to O_{\tilde{L}}((X))^\times$$

called as the *Coleman norm operator,* satisfying

$$\mathcal{N}_h g \circ h(X) = \prod_{\omega \in W_h^1} g(X[\underset{h}{+}]\omega)$$

for every $g \in O_{\tilde{L}}((X))^\times$, where $X[\underset{h}{+}]\omega \in \mathfrak{p}_{\tilde{L}^{sep}}[[X]]$ is defined by the addition on the $\mathfrak{p}_{\tilde{L}^{sep}}$-valued points of the formal group $F_h(X,Y)$.

Lemma 2.4 (Coleman).
(a) $\mathcal{N}_h : O_{\tilde{L}}((X))^\times \to O_{\tilde{L}}((X))^\times$ *is a multiplicative group homomorphism.*
(b) $\mathcal{N}_h g \equiv \phi_L g \pmod{\pi}$*, for every* $g \in O_{\tilde{L}}((X))^\times$.
(c) $\mathcal{C}(\tilde{L}, h) = \{g \in O_{\tilde{L}}((X))^\times : \mathcal{N}_h g = \phi_L g\}$ *is a subgroup of* $O_{\tilde{L}}((X))^\times$*, called the group of Coleman power series.*
(d) Reduction modulo π *induces an isomorphism*

$$\nabla_{\pi,h} : \mathcal{C}(\tilde{L}, h) \xrightarrow{\sim} \overline{\mathbb{F}}_{q_L}((X))^\times$$

defined by

$$\nabla_{\pi,h} : g \mapsto g(mod\ \pi)$$

for every $g \in \mathcal{C}(\tilde{L}, h)$.
(e) For $\boldsymbol{\omega} \in \Omega_h, g \in \mathcal{C}(\tilde{L}, h)$,

$$\{\beta_i = (\phi_L^{1-i} g)(\omega_i)\}_{1 \leq i \in \mathbb{Z}}$$

is a norm-compatible sequence in the tower $\{\overline{L^i}\}_{1 \leq i \in \mathbb{Z}}$ *of field extensions over* $\tilde{L}$*, which in return defines an isomorphism (depending on* $\boldsymbol{\omega}$*)*

$$\Delta_{\boldsymbol{\omega}} : \mathcal{C}(\tilde{L}, h) \xrightarrow{\sim} \varprojlim_i \ (\overline{L^i})^\times$$

given by

$$\Delta_{\boldsymbol{\omega}} : g \mapsto (\phi_L^{1-i} g(\omega_i))_{1 \leq i \in \mathbb{Z}}$$

for every $g \in \mathcal{C}(\tilde{L}, h)$*, where the projective limit* $\varprojlim_i (\overline{L^i})^\times$ *is defined with respect to the norm maps* $N_{\overline{L^j}/\overline{L^i}} : (\overline{L^j})^\times \to (\overline{L^i})^\times$ *for* $1 \leq i \leq j \in \mathbb{Z}$*.*

(f) The isomorphism $\Delta_{\boldsymbol{\omega}} : \mathcal{C}(\tilde{L}, h) \xrightarrow{\sim} \varprojlim_i (\overline{L^i})^\times$ *constructed in part (e) reduces to an isomorphism* $\Delta_{\boldsymbol{\omega}}|_{\mathcal{C}^\circ(\tilde{L},h)} : \mathcal{C}^\circ(\tilde{L}, h) \xrightarrow{\sim} \varprojlim_i U(\overline{L^i}) = \mathcal{U}(\tilde{L})$ *on the subgroup* $\mathcal{C}^\circ(\tilde{L}, h) = \mathcal{C}(\tilde{L}, h) \cap O_{\tilde{L}}[[X]]^\times$ *of Coleman power series of degree 0 in* $\mathcal{C}(\tilde{L}, h)$*.*

Suppose that K' is a *compatible extension* over K with respect to the fixed Lubin-Tate splitting (K, ϕ) (that is, K' is a finite extension over K and $K' \subset K_{\phi^{f(K'/K)}}$). Thus, $\phi' = \phi^{f(K'/K)}$ is a Lubin-Tate splitting for K' that will be fixed in the remaining of the text. There exists a natural morphism

$$M_{\phi, K'/K} : \mathfrak{G}(K', \phi') \to \mathfrak{G}(K, \phi)$$

called the 2-*abelian norm map* from $\mathfrak{G}(K', \phi')$ to $\mathfrak{G}(K, \phi)$, which is defined by the commutative squares

$$\begin{array}{ccc} \mathfrak{G}(K', \phi') & \xrightarrow{M_{\phi, K'/K}} & \mathfrak{G}(K, \phi) \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} \\ \widehat{K'^\times} & \xrightarrow{N_{K'/K}} & \widehat{K^\times} \end{array} \tag{2.3}$$

and

$$\begin{array}{ccc} \mathfrak{G}(K', \phi') & \xrightarrow{M_{\phi, K'/K}} & \mathfrak{G}(K, \phi) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2} \\ \overline{\kappa}_{K'}[[X]]^\times & \xrightarrow{N^{Coleman}_{K'/K}} & \overline{\kappa}_K[[X]]^\times. \end{array} \tag{2.4}$$

Here, the bottom horizontal arrow $N^{Coleman}_{\phi,K'/K} : \overline{\kappa}_{K'}[[X]]^{\times} \to \overline{\kappa}_K[[X]]^{\times}$ is defined via Coleman theory (Lemma 2.4) by the commutative diagram

$$\begin{array}{ccc} \mathcal{U}(\widetilde{K'}) & \xrightarrow{N_{\phi,K'/K}} & \mathcal{U}(\widetilde{K}) \\ {\scriptstyle \Delta_{\omega'}}\uparrow & & \uparrow{\scriptstyle \Delta_{\omega}} \\ \mathcal{C}^{\circ}(\widetilde{K'}, f_{\phi',K'}) & & \mathcal{C}^{\circ}(\widetilde{K}, f_{\phi,K}) \\ {\scriptstyle \nabla_{\pi_{K'}, f_{\phi',K'}}}\downarrow & & \downarrow{\scriptstyle \nabla_{\pi_K, f_{\phi,K}}} \\ \overline{\kappa}_{K'}[[X]]^{\times} & \xrightarrow{N^{Coleman}_{K'/K}} & \overline{\kappa}_K[[X]]^{\times}, \end{array} \tag{2.5}$$

where $\mathcal{U}(\widetilde{K})$ is defined to be the projective limit $\varprojlim_i U(\overline{K^{nr}K_i})$ (recall that, $K^{nr}K_i = K^i$ is the abelian extension of K which is class field to $U^i(K)$) taken over the norm maps $N_{\overline{K^{nr}K_j}/\overline{K^{nr}K_i}} : \overline{K^{nr}K_j} \to \overline{K^{nr}K_i}$ for $i \leq j, \mathcal{C}^{\circ}(\widetilde{K}, f_{\phi,K})$ denotes the Coleman power series of degree 0, the vertical arrows in the commutative diagram (2.5)

$$\mathcal{U}(\widetilde{K}) \xleftarrow{\Delta_{\omega}} \mathcal{C}^{\circ}(\widetilde{K}, f_{\phi,K}) \xrightarrow{\nabla_{\pi_K, f_{\phi,K}}} \overline{\kappa}_K[[X]]^{\times}$$

are the isomorphisms of Coleman theory (here $\boldsymbol{\omega} \in \Omega_{f_{\phi,K}}$, which is norm-compatible by part (e) of Lemma 2.3, is in particular chosen as $\omega_i = \pi_{K_i}$, for every $1 \leq i \in \mathbb{Z}$ by part (b) of Lemma 2.3), and

$$N_{\phi,K'/K} : \mathcal{U}(\widetilde{K'}) \to \mathcal{U}(\widetilde{K})$$

is the mapping defined by

$$N_{\phi,K'/K} : \{u'_m\} \mapsto \{u_n\},$$

with

$$u_n = \prod_{0 \leq \ell \leq f(K'/K)-1} \phi^{\ell}(\widetilde{N}_{K'_m/K_n}(u'_m))$$

for $m \gg n$ (that is, m is chosen sufficiently larger than n to ensure that $K_n \subseteq K'_m$).

The 2-abelian norm map $M_{\phi,K'/K} : \mathfrak{G}(K', \phi') \to \mathfrak{G}(K, \phi)$ has the following properties:

Lemma 2.5 (Koch-de Shalit).
(a) $M_{\phi,K'/K}(\mathfrak{G}(K',\phi')) = M_\phi(K'/K)$ *is a closed subgroup of finite index in* $\mathfrak{G}(K,\phi)$*;*
(b) if $K \subseteq K' \subseteq K''$ *is a tower of compatible extensions over* K *with respect to* (K,ϕ)*, the 2-abelian norm map*

$$M_{\phi,K''/K} : \mathfrak{G}(K'',\phi^{f(K''/K)}) \to \mathfrak{G}(K,\phi)$$

factors through

$$M_{\phi,K''/K} : \mathfrak{G}(K'',\phi'^{f(K''/K')}) \xrightarrow{M_{\phi',K''/K'}} \mathfrak{G}(K',\phi^{f(K'/K)}) \xrightarrow{M_{\phi,K'/K}} \mathfrak{G}(K,\phi).$$

If K'/K is an infinite algebraic extension defined as a union of compatible extensions E/K with respect to the Lubin-Tate splitting (K,ϕ), then put

$$M_\phi(K'/K) = \bigcap_E M_\phi(E/K)$$

which is a closed subgroup of $\mathfrak{G}(K,\phi)$.

§3. 2-abelian local class field theory.

The 2-abelian local class field theory states the following :

2-abelian local class field theory (Koch-de Shalit). *Let* K *be a (non-archimedean) local field, and with fixed Lubin-Tate splitting* ϕ *over* K*.*
1° There exists an order preserving (equivalently, inclusion reversing) bijection

$$L/K \leftrightarrow M_\phi(L/K)$$

between the set of all 2-abelian extensions of K *and the set of all closed subgroups of* $\mathfrak{G}(K,\phi)$*.*
2° (2-abelian local Artin map) There exists a canonical isomorphism

$$(?,K)_\phi = \iota_\phi : \mathfrak{G}(K,\phi) \stackrel{\sim}{\to} \ Gal\ (K^{(ab)^2}/K)$$

such that, for any 2-abelian extension L *over* K*, the surjective homomorphism*

$$(?,L/K)_\phi : \mathfrak{G}(K,\phi) \stackrel{\iota_\phi}{\to} (Gal\ (K^{(ab)^2}/K) \xrightarrow{res_{K^{(ab)^2}/L}} Gal\ (L/K)$$

has kernel $M_\phi(L/K)$, and thereby induces a canonical isomorphism

$$\iota_{\phi,L} : \mathfrak{G}(K,\phi)/M_\phi(L/K) \xrightarrow{\sim} \textit{Gal}\ (L/K).$$

3° (Functoriality) If K' is a compatible extension of K with respect to (K,ϕ), then the square

$$\begin{array}{ccc} \mathfrak{G}(K',\phi') & \xrightarrow{\iota_{\phi'}} & \textit{Gal}\ (K'^{(ab)^2}/K') \\ {\scriptstyle M_{K'/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{K'^{(ab)2}/K^{(ab)2}}} \\ \mathfrak{G}(K,\phi) & \xrightarrow{\iota_{\phi}} & \textit{Gal}\ (K^{(ab)^2}/K) \end{array}$$

is commutative

Note that, projection on the 1^{st}-component recovers the classical (abelian) local class field theory for K. That is, for a 2-abelian extension L/K, $\mathrm{proj}_1\ (M_\phi(L/K)) = N(L/K)$, and the abelian extension of K which is class field to $N(L/K)$ is the maximal abelian sub-extension in L/K.

The 2-abelian Artin map $\iota_\phi : \mathfrak{G}(K,\phi) \xrightarrow{\sim} \mathrm{Gal}(K^{(ab)^2}/K)$ is constructed in two steps.

Step 1. Let L be a finite Galois extension over K, $\phi_L \in \mathrm{Gal}\ (L^{nr}/L)$ the Frobenius automorphism over L, and let π be a fixed prime element in $\widetilde{L}$ such that $\nu_L(\pi) = 1$, where ν_L is the normalized valuation on L. Introduce the group

$$G(L/K,\pi) = \left\{ (\gamma,b) : \gamma \in \mathrm{Gal}(L^{nr}/K), b \in U(\widetilde{L})\text{s.t.}\frac{\phi_L(b)}{b} = \frac{\gamma(\pi)}{\pi} \right\}$$

with the law of composition defined by

$$(\gamma_1,b_1)(\gamma_2,b_2) = (\gamma_1\gamma_2, b_1\gamma_1(b_2)) \tag{3.1}$$

for every $(\gamma_1,b_1),(\gamma_2,b_2) \in G(L/K,\pi)$. The inverse $(\gamma,b)^{-1}$ of $(\gamma,b) \in G(L/K,\pi)$ is given by $(\gamma,b)^{-1} = (\gamma^{-1},\gamma^{-1}(b^{-1}))$.

Suppose that $(K \subseteq L \subseteq)L'$ is a finite Galois extension over K. Since $N_{\widetilde{L'}/\widetilde{L}} : \widetilde{L}'^{\times} \to \widetilde{L}^{\times}$ is a surjection, there exists a prime element $\pi' \in \widetilde{L}'$ such that $N_{\widetilde{L'}/\widetilde{L}}(\pi') = \pi$. There exists a surjective group homomorphism

$$\mathbf{r}_{L'/L} : G(L'/K,\pi') \to G(L/K,\pi) \tag{3.2}$$

defined by

$$\mathbf{r}_{L'/L} : (\gamma', b') \mapsto \left(\gamma'|_{L^{nr}}, \prod_{0 \le i \lneq d} \phi_L^i (N_{\widetilde{L'}/\widetilde{L}}(b')) \right)$$

for $(\gamma', b') \in G(L'/K, \pi')$, where $d = [L' \cap L^{nr} : L]$. The homomorphism (3.2) is compatible in towers of finite Galois extensions over K. That is, if the tower $K \subseteq L \subseteq L' \subseteq L''$ is all finite and Galois over K, and if π'' is a prime element in $\widetilde{L}''$ such that $N_{\widetilde{L''}/\widetilde{L'}}(\pi'') = \pi'$, then the homomorphism $\mathbf{r}_{L''/L} : G(L''/K, \pi'') \to G(L/K, \pi)$ factors through

$$\mathbf{r}_{L''/L} : G(L''/K, \pi'') \xrightarrow{\mathbf{r}_{L''/L'}} G(L'/K, \pi') \xrightarrow{\mathbf{r}_{L'/L}} G(L/K, \pi).$$

Given a norm-compatible sequence of prime elements $\boldsymbol{\omega} = \{\omega_i \in \overline{L^i} : N_{\overline{L^1}/\widetilde{L}}(\omega_1) = \pi\}$ in the tower $\{\overline{L^i}\}_{1 \le i \in \mathbb{Z}}$ of extensions over $\widetilde{L}$, there exists a normic Lubin-Tate power series $h(X) \in O_{\widetilde{L}}[[X]]$ belonging to π for $\widetilde{L}$ such that $\boldsymbol{\omega} \in \Omega_h$. Let $F_h(X,Y) \in O_{\widetilde{L}}[[X,Y]]$ be the unique Lubin-Tate formal group law over $O_{\widetilde{L}}$ attached to $h(X)$. If $(\gamma, b) \in G(L/K, \pi)$, then $\frac{\phi_L(b)}{b} = \frac{\gamma(\pi)}{\pi}$, so there exists a unique isomorphism $\eta = [b]_{h,\gamma h} : F_h \xrightarrow{\sim} F_{\gamma h}$ over $O_{\widetilde{L}}$ of the form $\eta(X) = bX+$ (higher-degree terms) $\in XO_{\widetilde{L}}[[X]]$ (note that, $\gamma h(X) \in XO_{\gamma(\widetilde{L})}[[X]]$ is a Lubin-Tate power series belonging to $\gamma(\pi)$ for $\gamma(\widetilde{L})$). Note that, since $\boldsymbol{\omega} \in \Omega_h$ (that is, $\omega_i \in \widetilde{W}^i_{\phi_L^{1-i}h}$ and $(\phi_L^{1-i}h)(\omega_i) = \omega_{i-1}), \omega'_i = \phi_L^{1-i}\eta(\omega_i) \in \mathfrak{p}_{\widetilde{L^{sep}}}$ is a primitive torsion point of level i on $F_{\phi_L^{1-i}\gamma h}$; that is, $\omega'_i \in \widetilde{W}^i_{\phi_L^{1-i}\gamma h}$, because for any $a \in \mathfrak{p}_L$,

$$\begin{aligned}
[a]_{\phi_L^{1-i}\gamma h}(\omega'_i) &= [a]_{\phi_L^{1-i}\gamma h}(\phi_L^{1-i}[b]_{h,\gamma h}(\omega_i)) \\
&= \phi_L^{1-i}([a]_{\gamma h}([b]_{h,\gamma h}(\omega_i))) \\
&= \phi_L^{1-i}([ba]_{h,\gamma h}(\omega_i)) \\
&= \phi_L^{1-i}([ab]_{h,\gamma h}(\omega_i)) \\
&= \phi_L^{1-i}([b]_{h,\gamma h}([a]_h(\omega_i))) \\
&= \phi_L^{1-i}[b]_{h,\gamma h}([a]_{\phi_L^{1-i}h}(\omega_i)),
\end{aligned}$$

which proves that $[a]_{\phi_L^{1-i}\gamma h}(\omega'_i) = 0$ for every $a \in \mathfrak{p}_L^i$ and $[c]_{\phi_L^{1-i}\gamma h}(\omega'_i) \neq 0$ for some $c \in \mathfrak{p}_L^{i-1}$ (since $b \in U(\widetilde{L})$). Moreover,

$$\begin{aligned}
\phi_L^{1-i}\gamma h(\omega'_i) &= \phi_L^{1-i}\gamma h(\phi_L^{1-i}\eta(\omega_i)) \\
&= \phi_L^{1-i}\gamma h(\phi_L^{1-i}[b]_{h,\gamma h}(\omega_i)) \\
&= \phi_L^{1-i}(\gamma h([b]_{h,\gamma h}(\omega_i))) \\
&= \phi_L^{1-i}(\gamma h \circ [b]_{h,\gamma h}(\omega_i)) \\
&= \phi_L^{1-i}((\phi_L[b]_{h,\gamma h} \circ h)(\omega_i)) \\
&= \phi_L^{1-(i-1)}[b]_{h,\gamma h}(\phi_L^{1-i}h(\omega_i)) \\
&= \phi_L^{1-(i-1)}[b]_{h,\gamma h}(\omega_{i-1}) = \omega'_{i-1},
\end{aligned}$$

proving that $\boldsymbol{\omega}' = (\omega'_i)_{1\leq i\in\mathbb{Z}} \in \Omega_{\gamma h}$. Now, there exists a unique extension of $\gamma \in \mathrm{Gal}(L^{nr}/K)$ to a K-automorphism $\gamma' : \bigcup_{1\leq i\in\mathbb{Z}} \overline{L^i} \to \bigcup_{1\leq i\in\mathbb{Z}} \overline{L^i}$ satisfying $\gamma'\boldsymbol{\omega} = \boldsymbol{\omega}'$. In fact, $\overline{L^i} = \widetilde{L}(\omega_i) = \widetilde{L}(\omega'_i)$, so the unique K-automorphism γ' of the field $\bigcup_{1\leq i\in\mathbb{Z}} \overline{L^i}$ is defined by the conditions

$$\gamma'|_{L^{nr}} = \gamma$$

and

$$\gamma' : \omega_i \mapsto \omega'_i = \phi_L^{1-i}[b]_{h,\gamma h}(\omega_i)$$

for $1 \leq i \in \mathbb{Z}$

The following result (Proposition 2.4 in [4]) is the 1^{st} fundamental fact, which is utilized in the construction of the metabelian local Artin map.

Proposition 3.1 (Koch-de Shalit). *Fix a norm-compatible sequence of prime elements*

$$\boldsymbol{\omega} = \{\omega_i \in \overline{L^i} : N_{\overline{L^1}/\widetilde{L}}(\omega_1) = \pi\}$$

in the tower $\{\overline{L^i}\}_{1\leq i\in\mathbb{Z}}$ *of extensions over* $\widetilde{L}$*, where* L^i *denotes the abelian extension of* L *which is class field to* $U^i(L)$ *and* $\overline{L^i}$ *its completion. Then, there exists an isomorphism (depending on* $\boldsymbol{\omega}$*)*

$$\iota_{\boldsymbol{\omega}} : G(L/K, \pi; \phi_L) \xrightarrow{\sim} \ Gal(L^{ab}/K),$$

where $\iota_{\boldsymbol{\omega}}(\gamma, b) \in \ Gal(L^{ab}/K)$ *is uniquely defined by the conditions*

$$\iota_{\boldsymbol{\omega}}(\gamma, b)|_{L^{nr}} = \gamma$$

and

$$\iota_{\boldsymbol{\omega}}(\gamma, b)(\omega_i) = \phi_L^{1-i}[b]_{h,\gamma h}(\omega_i)$$

for $1 \leq i \in \mathbb{Z}$ *(here,* $h \in O_{\widetilde{L}}[[X]]$ *is the unique normic Lubin-Tate formal power series belonging to* π *for* $\widetilde{L}$ *such that* $\boldsymbol{\omega} \in \Omega_h$*), which makes the following diagram commutative:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U(L) & \longrightarrow & G(L/K, \pi; \phi_L) & \longrightarrow & Gal(L^{nr}/K) & \longrightarrow & 1 \\
 & & \Big\downarrow {\scriptstyle (?,L)^{-1}} & & \Big\downarrow {\scriptstyle \iota,\omega} & & \Big\| & & \\
1 & \longrightarrow & Gal(L^{ab}/L^{nr}) & \longrightarrow & Gal(L^{ab}/K) & \longrightarrow & Gal(L^{nr}/K) & \longrightarrow & 1.
\end{array}$$

Proof. For a proof, look at (Proposition 2.4, [4]). □

Remark 3.2. Since we have fixed a Lubin-Tate splitting ϕ over K, we have a unique prime element $\pi_L \in \widetilde{L}$ (such that $\pi_L \in \overline{\wp}_\phi$) by part (a) of Lemma 2.3 satisfying $\nu_L(\pi_L) = \nu_K(N_{\widetilde{L}/\widetilde{K}}(\pi_L)) = \nu_K(\pi_K) = 1$, and the norm-compatible sequence of prime elements $\boldsymbol{\omega}$ considered in Proposition 3.1 is uniquely determined by $\omega_i = \pi_{L_i}$, since L^i/L_i is an unramified extension and by part (a) of Lemma 2.3 (that is, $\boldsymbol{\omega} \subseteq \overline{\wp}_\phi$). Thus, by Proposition 3.1, the isomorphism

$$\iota_{\boldsymbol{\omega}} : G(L/K, \pi_L) \xrightarrow{\sim} \mathrm{Gal}(L^{ab}/K)$$

is uniquely determined by the Lubin-Tate splitting (K, ϕ), which is then denoted by

$$\iota_{\phi, L/K} : G(L/K, \pi_L) \xrightarrow{\sim} \mathrm{Gal}(L^{ab}/K).$$

Suppose that $K \subseteq L \subseteq L'$ are finite Galois extensions over K, then the following square

$$\begin{array}{ccc} G(L'/K, \pi_{L'}) & \xrightarrow{\mathbf{r}_{L'/L}} & G(L/K, \pi_L) \\ {\scriptstyle \iota_{\phi,L'/K}} \Big\downarrow & & \Big\downarrow {\scriptstyle \iota_{\phi,L/K}} \\ \mathrm{Gal}((L')^{ab}/K) & \xrightarrow{\mathrm{res}_{(L')^{ab}/L^{ab}}} & \mathrm{Gal}(L^{ab}/K) \end{array} \tag{3.3}$$

is commutative.

Step 2. The second step consists of constructing a canonical isomorphism

$$\iota_{\phi,d} : \mathfrak{G}_d(K, \phi) \xrightarrow{\sim} \mathrm{Gal}((K_\infty K_d^{nr})^{ab}/K)$$

which will be defined below as follows. Consider for $1 \leq i \in \mathbb{Z}$, the field $K_i K_d^{nr}$. For $(a, \xi) \in \mathfrak{G}_d(K, \phi)$, let $\gamma_i = (a, K)^{-1}|_{K_i K^{nr}} \in \mathrm{Gal}(K^i/K)$ (recall that $K_i K^{nr} = K^i$), and let $\beta_{i,\xi} = \phi^{1-i} \nabla^{-1}_{\pi_K, f_{\phi,K}}(\xi)(\pi_{K^i}) \in U(\overline{K^i})$, where $\nabla^{-1}_{\pi_K, f_{\phi,K}}(\xi) = g_\xi \in \mathcal{C}^\circ(\widetilde{K}, f_{\phi,K})$ is the unique Coleman power series of degree 0 lifting $\xi \in \overline{\kappa}_K[[X]]^\times$ (c.f part (d) of Lemma 2.4), and $\{\beta_{i,\xi}\}_{1 \leq i \in \mathbb{Z}}$ is a norm-compatible sequence in the tower $\{U(\overline{K^i})\}_{1 \leq i \in \mathbb{Z}}$ (c.f parts (e) and (f) of Lemma 2.4). Observe that, $\gamma_i \in \mathrm{Gal}((K_i K_d^{nr})^{nr}/K)$ and $\beta_{i,\xi} \in U(\widetilde{K_i K_d^{nr}})$ (note that $(K_i K_d^{nr})^{nr} = K^i$ and $\widetilde{K_i K_d^{nr}} = \overline{K^i}$) satisfies

$$\begin{aligned} \frac{\phi_{K_i K_d^{nr}}(\beta_{i,\xi})}{\beta_{i,\xi}} &= \frac{\phi^d(\beta_{i,\xi})}{\beta_{i,\xi}} \\ &= \frac{\phi^d(\phi^{1-i} g_\xi(\pi_{K^i}))}{\phi^{1-i} g_\xi(\pi_{K^i})} \\ &= \phi^{1-i}\left(\frac{\phi^d g_\xi(\pi_{K^i})}{g_\xi(\pi_{K^i})}\right) \\ &= \phi^{1-i}\left(\frac{\phi^d g_\xi}{g_\xi}(\pi_{K^i})\right) = \phi^{1-i}\left(\frac{[u]_{f_{\phi,K}}}{X}(\pi_{K^i})\right) \end{aligned}$$

by the fact that $\frac{\phi^d \xi}{\xi} = \frac{\{u\}_{f_{\phi,K}}}{X}$ (where $a = u\pi_K^\nu \in \widehat{K^\times}$ for some $u \in U(K)$ and $\nu \in \widehat{\mathbb{Z}}$) and by part (d) of Lemma 2.4. Therefore (recall that $\pi_{K^i} = \pi_{K_i K_d^{nr}} = \pi_{K_i} = \omega_i$),

$$\phi^{1-i}\left(\frac{[u]_{f_{\phi,K}}}{X}(\pi_{K^i})\right) = \frac{[u]_{\phi^{1-i}f_{\phi,K}}(\phi^{1-i}\omega_i)}{\phi^{1-i}\omega_i}$$
$$= \frac{[u]_{\phi^{1-i}f_{\phi,K}}(\omega_i)}{\omega_i}$$
$$= \frac{(u^{-1},K)(\omega_i)}{\omega_i} = \frac{\gamma_i(\pi_{K_iK_d^{nr}})}{\pi_{K_iK_d^{nr}}},$$

by the fact that $K_i \subset K_\phi$ and by eq. no. (2.1), proving that $(\gamma_i, \beta_{i,\xi}) \in G(K_iK_d^{nr}/K, \pi_{K_iK_d^{nr}})$.

The following result (Proposition 2.13 in [4]) is the 2^{nd} fundamental fact that is utilized in the construction of the metabelian local Artin map.

Proposition 3.3 (Koch-de Shalit). *The mapping*

$$\iota_{\phi,d} : \mathfrak{G}_d(K,\phi) \to \ Gal((K_\infty K_d^{nr})^{ab}/K)$$

defined by

$$\iota_{\phi,d} : (a,\xi) \mapsto \varprojlim_i \ \iota_{\phi,K_iK_d^{nr}/K}(\gamma_i, \beta_{i,\xi})$$

for every $(a,\xi) \in \mathfrak{G}_d(K,\phi)$, *where* $\gamma_i = (a^{-1},K)|_{K_iK^{nr}} \in Gal(K^i/K)$ *and* $\beta_{i,\xi} = \phi^{1-i}\nabla^{-1}_{\pi_K,f_{\phi,K}}(\xi)(\pi_{K^i}) \in U(\overline{K^i})$ *for* $1 \le i \in \mathbb{Z}$, *is an isomorphism. Here, the map*

$$\varprojlim_i \ \iota_{\phi,K_iK_d^{nr}/K} : \ \varprojlim_i \ G(K_iK_d^{nr}/K, \pi_{K_i}) \xrightarrow{\sim} \underbrace{\varprojlim_i \ Gal((K_iK_d^{nr})^{ab}/K)}_{Gal((K_\infty K_d^{nr})^{ab}/K)}$$

is the isomorphism defined via the commutative square eq. no. (3.3).

Proof. For a proof, look at (Proposition 2.13, [4]). □

Note that, the square

$$\begin{array}{ccc} \mathfrak{G}_{d_2}(K,\phi) & \xrightarrow{\iota_{\phi,d_2}} & \mathrm{Gal}((K_\infty K_{d_2}^{nr})^{ab}/K) \\ {\scriptstyle \tau(\phi)_{d_1}^{d_2}}\downarrow & & \downarrow {\scriptstyle res_{(K_\infty K_{d_2}^{nr})^{ab}/(K_\infty K_{d_1}^{nr})^{ab}}} \\ \mathfrak{G}_{d_1}(K,\phi) & \xrightarrow{\iota_{\phi,d_1}} & \mathrm{Gal}((K_\infty K_{d_1}^{nr})^{ab}/K) \end{array} \tag{3.4}$$

is commutative for $d_1|d_2$. Now, the 2-abelian local Artin map

$$\iota_\phi : \mathfrak{G}(K,\phi) \xrightarrow{\sim} \mathrm{Gal}(K^{(ab)^2}/K)$$

is defined by passing to the projective limits defined via the commutative square eq. no. (3.4) as

$$\iota_\phi = \varprojlim_d \iota_{\phi,d} : \varprojlim_d \mathfrak{G}_d(K,\phi) = \mathfrak{G}(K,\phi) \xrightarrow{\sim} \mathrm{Gal}(K^{(ab)^2}/K) = \varprojlim_d \mathrm{Gal}((K_\infty K_d^{nr})^{ab}/K).$$

PART II: GALOIS CONJUGATION LAW

§4. Preliminaries

Let $\sigma : K \hookrightarrow K^{sep}$ be any embedding of K into K^{sep}, and fix once and for all an extension $\widetilde{\sigma} \in \mathrm{Aut}(K^{sep})$ of $\sigma : K \hookrightarrow K^{sep}$ to K^{sep}. Let K' be a finite extension over K, always assumed to be a subfield of K^{sep}. Since $\widetilde{\sigma}(K'^{nr}) = (\widetilde{\sigma}K')^{nr}$, by continuity, $\widetilde{\sigma}(\widetilde{K'}) = \widetilde{\widetilde{\sigma}(K')}$. Thus, there is an isomorphism

$$\widetilde{\sigma} : O_{\widetilde{K'}}((X)) \xrightarrow{\sim} O_{\widetilde{\widetilde{\sigma}(K')}}((X)),$$

which is defined by the application of $\widetilde{\sigma}$ on the coefficients of the formal Laurent series in $O_{\widetilde{K'}}((X))$, and there exists a unique isomorphism $\widetilde{\sigma}^* : \overline{\kappa}_{K'} \xrightarrow{\sim} \overline{\kappa}_{\widetilde{\sigma}(K')}$ extending to

$$\widetilde{\sigma}^* : \overline{\kappa}_{K'}((X)) \xrightarrow{\sim} \overline{\kappa}_{\widetilde{\sigma}(K')}((X))$$

and making the square

$$\begin{array}{ccc}
O_{\widetilde{K'}}((X)) & \xrightarrow{mod \mathfrak{p}_{\widetilde{K'}}} & \overline{\kappa}_{K'}((X)) \\
\widetilde{\sigma} \downarrow & & \downarrow \exists ! \widetilde{\sigma}^{*} \\
O_{\widetilde{\sigma(K')}}((X)) & \xrightarrow{mod \mathfrak{p}_{\widetilde{\sigma(K')}}} & \overline{\kappa}_{\widetilde{\sigma(K')}}((X))
\end{array} \tag{4.1}$$

commutative. Moreover, $\overline{(K')^{sep}} = \overline{K^{sep}}$ is algebraically closed, so $\widetilde{K'}^{sep} \subseteq \overline{K^{sep}}$, which shows that $\widetilde{\sigma} \in \mathrm{Aut}(K^{sep})$ has a unique extension to an automorphism of $\overline{K^{sep}}$ (and hence to $\widetilde{K'}^{sep}$) by continuity, which will again be denoted by $\widetilde{\sigma} : \overline{K^{sep}} \to \overline{K^{sep}}$. The following simple observation is fundamental in what follows:

Basic Observation. *If ϕ' is a Lubin-Tate splitting over K'; that is, ϕ' is a fixed automorphism of K'^{sep} such that $\phi'|_{K'^{nr}}$ is the Frobenius automorphism $\phi_{K'}$ over K'; then $\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1}$ is the Frobenius automorphism over $\widetilde{\sigma}K'$ and $\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1}$ is a Lubin-Tate splitting over $\widetilde{\sigma}(K')$.*

In view of this basic observation, there exists a bijection between the set of all Lubin-Tate splittings over K' and the set of all Lubin-Tate splittings over $\widetilde{\sigma}(K')$ defined by

$$\phi' \longleftrightarrow \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1}.$$

If π' is a prime element in $\widetilde{K'}$, then $\widetilde{\sigma}(\pi')$ is a prime element in $\widetilde{\sigma}(\widetilde{K'}) = \widetilde{\sigma(K')}$. Moreover, if $\nu_{K'}(\pi') = 1$, where $\nu_{K'}$ is the normalized valuation on K', then $\nu_{\widetilde{\sigma(K')}}(\widetilde{\sigma}(\pi')) = 1$, since $\nu_{\widetilde{\sigma(K')}}(\widetilde{\sigma}(\pi')) = \nu_{K'}(\pi')$. If $h(X) \in O_{\widetilde{K'}}[[X]]$ is a Lubin-Tate power series belonging to π' for $\widetilde{K'}$, then $\widetilde{\sigma}h(X) \in O_{\widetilde{\sigma(K')}}[[X]]$ is a Lubin-Tate power series belonging to the prime $\widetilde{\sigma}(\pi')$ for $\widetilde{\sigma(K')}$; that is, $\widetilde{\sigma}\mathcal{F}'_{\pi'} = \mathcal{F}'_{\widetilde{\sigma}(\pi')}$. Furthermore, $\widetilde{\sigma}F_h(X,Y) = F_{\widetilde{\sigma}h}(X,Y) \in O_{\widetilde{\sigma(K')}}[[X,Y]]$ is the unique Lubin-Tate formal group law over $O_{\widetilde{\sigma(K')}}$ attached to the Lubin-Tate power series $\widetilde{\sigma}h(X)$, which clearly satisfies

$$(\phi_{\widetilde{\sigma(K')}}F_{\widetilde{\sigma}h})(\widetilde{\sigma}h(X), \widetilde{\sigma}h(Y)) = (\widetilde{\sigma}h)(F_{\widetilde{\sigma}h}(X,Y)).$$

In fact,

$$(\widetilde{\sigma}h)(F_{\widetilde{\sigma}h}(X,Y)) = (\widetilde{\sigma}h)(\widetilde{\sigma}F_h(X,Y))$$

$$
\begin{aligned}
&= \widetilde{\sigma}(h(F_h(X,Y))) \\
&= \widetilde{\sigma}((\phi_{K'}F_h)(h(X),h(Y))) \\
&= ((\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1})F_{\widetilde{\sigma}h})(\widetilde{\sigma}h(X),\widetilde{\sigma}h(Y)) \\
&= (\phi_{\widetilde{\sigma}(K')}F_{\widetilde{\sigma}h})(\widetilde{\sigma}h(X),\widetilde{\sigma}h(Y)),
\end{aligned}
$$

proving the desired equality. The endomorphism ring $\mathrm{End}_{O_{\widetilde{\sigma(K')}}}(F_{\widetilde{\sigma}h}(X,Y))$ over $O_{\widetilde{\sigma(K')}}$ is isomorphic to the ring of integers $O_{\widetilde{\sigma}(K')}$ of $\widetilde{\sigma}(K')$ under the isomorphism

$$\widetilde{\sigma}(a) \mapsto [\widetilde{\sigma}(a)]_{\widetilde{\sigma}h} = \widetilde{\sigma}[a]_h,$$

for every $a \in O_{K'}$. In fact,

$$
\begin{aligned}
\widetilde{\sigma}[a]_h(F_{\widetilde{\sigma}h}(X,Y)) &= \widetilde{\sigma}([a]_h(F_h(X,Y))) \\
&= \widetilde{\sigma}(F_h([a]_h(X),[a]_h(Y))) \\
&= F_{\widetilde{\sigma}h}(\widetilde{\sigma}[a]_h(X),\widetilde{\sigma}[a]_h(Y)),
\end{aligned}
$$

and $\widetilde{\sigma}[a]_h = \widetilde{\sigma}aX+$ (higher-degree terms) $\in XO_{\widetilde{\sigma(K')}}[[X]]$, proving that $\widetilde{\sigma}[a]_h = [\widetilde{\sigma}a]_{\widetilde{\sigma}h}$ by the uniqueness. Now, suppose that π'_o is another prime element of $\widetilde{K'}$, $h_o(X) \in \mathcal{F}'_{\pi'_o}$ and $0 \neq a \in O_{\widetilde{K'}}$ such that $\frac{\phi_{K'}(a)}{a} = \frac{\pi'_0}{\pi'}$ (therefore, there exists a unique homomorphism $[a]_{h,h_o} : F_h \to F_{h_o}$ of the form $[a]_{h,h_o}(X) = aX+$ (higher-degree terms) $\in XO_{\widetilde{K'}}[[X]]$); then

$$\frac{(\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1})(\widetilde{\sigma}a)}{\widetilde{\sigma}(a)} = \widetilde{\sigma}\left(\frac{\phi_{K'}a}{a}\right) = \widetilde{\sigma}\left(\frac{\pi'_o}{\pi'}\right) = \frac{\widetilde{\sigma}(\pi'_o)}{\widetilde{\sigma}(\pi')},$$

and $\widetilde{\sigma}[a]_{h,h_o} = [\widetilde{\sigma}a]_{\widetilde{\sigma}h,\widetilde{\sigma}h_o} : F_{\widetilde{\sigma}h} \to F_{\widetilde{\sigma}h_o}$ is the unique homomorphism from $F_{\widetilde{\sigma}h}$ to $F_{\widetilde{\sigma}h_o}$ of the form $\widetilde{\sigma}aX+$ (higher-degree terms) $\in XO_{\widetilde{\sigma(K')}}[[X]]$. In particular, if $x \in W^i_h$, then

$$0 = \widetilde{\sigma}([a]_h(x)) = \widetilde{\sigma}[a]_h(\widetilde{\sigma}x) = [\widetilde{\sigma}a]_{\widetilde{\sigma}h}(\widetilde{\sigma}x)$$

for every $a \in \mathfrak{p}^i_{K'}$, proving that $\widetilde{\sigma}W^i_h = W^i_{\widetilde{\sigma}h}$; and if $\boldsymbol{\omega} = (\omega_1, \omega_2, ...) \in \Omega_h$, then

$$\widetilde{\sigma}\omega_i \in \widetilde{\sigma}\widetilde{W}^i_{\phi^{1-i}_{K'}h} = \widetilde{W}^i_{(\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1})^{1-i}\widetilde{\sigma}h}$$

and

$$((\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1})^{1-i}\widetilde{\sigma}h)(\widetilde{\sigma}\omega_i) = \widetilde{\sigma}\omega_{i-1}$$

for $1 \leq i \in \mathbb{Z}$, proving that $\widetilde{\sigma}\Omega_h = \Omega_{\widetilde{\sigma}h}$.

The following observation will be central in what follows.

Lemma 4.1. *Consider the Lubin-Tate splitting $\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}$ over σK. Clearly, $(\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \widetilde{\sigma}K_\phi$, and*

(a) the unique norm-compatible sequence of primes

$$\wp_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \{\pi_E \in E^\times : \sigma K \subseteq E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}; [E : \sigma K] < \infty\}$$

is given by $\wp_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \widetilde{\sigma}(\wp_\phi)$;

(b) for any $E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}, [E : \sigma K] < \infty$, the unique Lubin-Tate formal power series $f_{\tilde{\sigma}\phi\tilde{\sigma}^{-1},E} \in O_E[[X]]$ belonging to π_E (chosen as in (a)) for E, and satisfying $f_{\tilde{\sigma}\phi\tilde{\sigma}^{-1},E}(\pi_{E_n}) = \pi_{E_{n-1}}$ for $n \geq 2, f_{\tilde{\sigma}\phi\tilde{\sigma}^{-1},E}(\pi_{E_1}) = 0$ is given by $f_{\tilde{\sigma}\phi\tilde{\sigma}^{-1},E} = \tilde{\sigma} f_{\phi,\tilde{\sigma}^{-1}E}$;

(c) the unique Lubin-Tate formal group law $F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}$ defined over O_E associated to the unique Lubin-Tate formal power series $f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} \in O_E[[X]]$ satisfying

$$F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} \circ (f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} \times f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}) = f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} \circ F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}$$

for each $\sigma K \subseteq E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}$ with $[E : \sigma K] < \infty$ is given by $F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} = \widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}E}$;

(d) for $a \in O_E$, the unique endomorphism $[a]_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}} : F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} \to F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}$ of the Lubin-Tate formal group law $F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}$ for each $\sigma K \subseteq E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}$ with $[E : \sigma K] < \infty$ is given by $[a]_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}} = \widetilde{\sigma}[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1},E}}$ and $\{a\}_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}} = \widetilde{\sigma}^*\{\widetilde{\sigma}^{-1}a\}_{f_{\phi,\widetilde{\sigma}^{-1}E}}$;

(e) $\widetilde{\sigma}\Omega_{f_{\phi,K}} = \Omega_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},K}}$ and every $\widetilde{\sigma}\boldsymbol{\omega} \in \Omega_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},K}}$ for $\boldsymbol{\omega} \in \Omega_{f_{\phi,K}}$ is norm-compatible.

Proof. The assertion $(\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \widetilde{\sigma}K_\phi$ follows from the fact that

$$(\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \{x \in K^{sep} | \phi(\widetilde{\sigma}^{-1}(x)) = \widetilde{\sigma}^{-1}(x)\}.$$

(a) Let $\wp_\phi = \{\pi_L \in L^\times : K \subseteq L \subseteq K_\phi; [L : K] < \infty\}$ be the unique norm-compatible sequence of prime elements corresponding to the Lubin-Tate splitting (K, ϕ). Consider $K \subseteq L \subseteq M \subseteq K_\phi$ with $[M : K] < \infty$. Let $\varepsilon_L(M, K^{sep})$ denote the (finite) set of all L-embeddings $M \hookrightarrow K^{sep}$. Then, clearly

$$\widetilde{\sigma}^{-1}\varepsilon_{\widetilde{\sigma}(L)}(\widetilde{\sigma}(M), K^{sep})\widetilde{\sigma} = \varepsilon_L(M, K^{sep}).$$

Thus, computing the norm $N_{\widetilde{M}/\widetilde{L}}(\pi_M)$ (which is equal to $\pi_L \in L^\times$),

$$\begin{aligned}\pi_L &= N_{\widetilde{M}/\widetilde{L}}(\pi_M)\\ &= N_{M/L}(\pi_M)\\ &= \prod_{\widetilde{\sigma}^{-1}\tau\widetilde{\sigma}}(\widetilde{\sigma}^{-1}\tau\widetilde{\sigma})(\pi_M) \qquad (\tau \text{ runs over } \varepsilon_{\widetilde{\sigma}(L)}(\widetilde{\sigma}(M), K^{sep}))\\ &= \widetilde{\sigma}^{-1}\prod_{\tau}\tau(\widetilde{\sigma}(\pi_M))\\ &= \widetilde{\sigma}^{-1}N_{\widetilde{\sigma}(M)/\widetilde{\sigma}(L)}(\widetilde{\sigma}(\pi_M))\\ &= \widetilde{\sigma}^{-1}N_{\widetilde{\widetilde{\sigma}(M)}/\widetilde{\widetilde{\sigma}(L)}}(\widetilde{\sigma}(\pi_M)),\end{aligned}$$

it follows that $\widetilde{\sigma}(\pi_L) = N_{\widetilde{\widetilde{\sigma}(M)}/\widetilde{\widetilde{\sigma}(L)}}(\widetilde{\sigma}(\pi_M))$, and the assertion follows.
(b) For $E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}$ with $[E : \sigma K] < \infty$, it suffices to prove that $\widetilde{\sigma}^{-1}(E)_n = \widetilde{\sigma}^{-1}(E_n)$ for $1 \leq n \in \mathbb{Z}$. In fact, $N_{E_n/E}E_n^\times =< \pi_E > U^n(E)$, since E_n is class-field to the subgroup $< \pi_E > U^n(E)$ of $E^\times$. Thus,

$$\begin{aligned}N_{\widetilde{\sigma}^{-1}(E_n)/\widetilde{\sigma}^{-1}(E)}\widetilde{\sigma}^{-1}(E_n)^\times &= \widetilde{\sigma}^{-1}N_{E_n/E}E_n^\times\\ &= < \widetilde{\sigma}^{-1}\pi_E > U^n(\widetilde{\sigma}^{-1}E)\\ &= < \pi_{\widetilde{\sigma}^{-1}E} > U^n(\widetilde{\sigma}^{-1}E) = N_{\widetilde{\sigma}^{-1}(E)_n/\widetilde{\sigma}^{-1}(E)}\widetilde{\sigma}^{-1}(E)_n^\times\end{aligned}$$

by part (a). Hence, by the local class field theory, $\widetilde{\sigma}^{-1}(E_n) = \widetilde{\sigma}^{-1}(E)_n$.
(c) Let $\sigma K \subseteq E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}$ with $[E : \sigma K] < \infty$. Let $F_{\phi,\widetilde{\sigma}^{-1}E}$ be the unique Lubin-Tate formal group law defined over $O_{\widetilde{\sigma}^{-1}(E)}$ associated to the Lubin-Tate formal power series $f_{\phi,\widetilde{\sigma}^{-1}(E)} \in O_{\widetilde{\sigma}^{-1}(E)}[[X]]$ satisfying

$$F_{\phi,\widetilde{\sigma}^{-1}(E)} \circ (f_{\phi,\widetilde{\sigma}^{-1}(E)} \times f_{\phi,\widetilde{\sigma}^{-1}(E)}) = f_{\phi,\widetilde{\sigma}^{-1}(E)} \circ F_{\phi,\widetilde{\sigma}^{-1}(E)}.$$

Then, $\widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)}$ is a Lubin-Tate formal group law over $O_E = \widetilde{\sigma}O_{\widetilde{\sigma}^{-1}(E)}$ satisfying

$$\widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)} \circ (\widetilde{\sigma}f_{\phi,\widetilde{\sigma}^{-1}(E)} \times \widetilde{\sigma}f_{\phi,\widetilde{\sigma}^{-1}(E)}) = \widetilde{\sigma}f_{\phi,\widetilde{\sigma}^{-1}(E)} \circ \widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)}.$$

Since $\widetilde{\sigma}f_{\phi,\widetilde{\sigma}^{-1}(E)} = f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}$ by part (b), it follows that $F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E} = \widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)}$.

(d) Let $\sigma K \subseteq E \subseteq (\sigma K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}$ with $[E : \sigma K] < \infty$. For $a \in O_E$, there exists a unique endomorphism

$$[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}} : F_{\phi,\widetilde{\sigma}^{-1}(E)} \to F_{\phi,\widetilde{\sigma}^{-1}(E)}$$

of the Lubin-Tate formal group law $F_{\phi,\widetilde{\sigma}^{-1}(E)}$ over $O_{\widetilde{\sigma}^{-1}(E)}$ of the form $[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}} = \widetilde{\sigma}^{-1}aX+$ (higher degree terms). Then, $\widetilde{\sigma}[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}} = aX+$ (higher degree terms) is an endomorphism of the Lubin-Tate formal group law $\widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)} = F_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}$ over O_E, since

$$\widetilde{\sigma}[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}} \circ \widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)} = \widetilde{\sigma}F_{\phi,\widetilde{\sigma}^{-1}(E)} \circ (\widetilde{\sigma}[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}} \times \widetilde{\sigma}[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}}).$$

Thus, $[a]_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}} = \widetilde{\sigma}[\widetilde{\sigma}^{-1}a]_{f_{\phi,\widetilde{\sigma}^{-1}(E)}}$ and $\{a\}_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},E}} = \widetilde{\sigma}^{*}\{\widetilde{\sigma}^{-1}a\}_{f_{\phi,\widetilde{\sigma}^{-1}(E)}}$.
(e) The first equality follows from part (b), and the second assertion follows from a norm computation similar as in the proof of part (a). □

The group of d-Koch-de Shalit vectors with respect to the Lubin-Tate splitting $\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}$ over σK is explicitely described by
$\mathfrak{G}_d(\sigma K, \widetilde{\sigma}\phi\widetilde{\sigma}^{-1})$

$$= \left\{ (b,\eta) : \begin{array}{c} b = \nu\pi_{\sigma K}^{\mu} \in \widehat{\sigma(K)}^{\times} \text{ s.t } \nu \in U(\sigma K), \pi_{\sigma K} = \sigma(\pi_K), \mu \in \hat{\mathbb{Z}} \\ \eta = \eta(X) \in \overline{\kappa}_{\sigma K}[[X]]^{\times} \text{ s.t } \frac{(\widetilde{\sigma}^{*}\phi^{d}(\widetilde{\sigma}^{*})^{-1})\eta}{\eta} = \frac{\{\nu\}_{f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma K}}}{X} \end{array} \right\},$$

and the group of Koch-de Shalit vectors with respect to the Lubin-Tate splitting $\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}$ over σK is then the projective limit $\mathfrak{G}(\sigma K, \widetilde{\sigma}\phi\widetilde{\sigma}^{-1}) = \varprojlim_{d} \mathfrak{G}_d(\sigma K, \widetilde{\sigma}\phi\widetilde{\sigma}^{-1})$ taken over the connecting morphisms $\tau(\widetilde{\sigma}\phi\widetilde{\sigma}^{-1})_{d_1}^{d_2} : \mathfrak{G}_{d_2}(\sigma K, \widetilde{\sigma}\phi\widetilde{\sigma}^{-1}) \to \mathfrak{G}_{d_1}(\sigma K, \widetilde{\sigma}\phi\widetilde{\sigma}^{-1})$ defined for any positive integers d_1, d_2 with $d_1 | d_2$.

Remark 4.2. If K' is a compatible extension over K with respect to the Lubin-Tate splitting (K, ϕ), then $\widetilde{\sigma}(K')$ is compatible over σK with respect to the Lubin-Tate splitting $(\sigma K, \widetilde{\sigma}\phi\widetilde{\sigma}^{-1})$.

Lemma 4.3.
(a) Let π' be a prime element of $\widetilde{K'}$, and let $h(X) \in \mathcal{F}'_{\pi'}$. Then the square

$$\begin{array}{ccc} O_{\widetilde{K'}}((X))^\times & \xrightarrow{\mathcal{N}_h} & O_{\widetilde{K'}}((X))^\times \\ \Big\downarrow \widetilde{\sigma} & & \Big\downarrow \widetilde{\sigma} \\ O_{\widetilde{\widetilde{\sigma}(K')}}((X))^\times & \xrightarrow{\mathcal{N}_{\widetilde{\sigma} h}} & O_{\widetilde{\widetilde{\sigma}(K')}}((X))^\times \end{array}$$

is commutative.
(b) $\mathcal{C}(\widetilde{K'}, h)$ is mapped isomorphically onto $\mathcal{C}(\widetilde{\widetilde{\sigma}(K')}, \widetilde{\sigma}h)$ under the isomorphism

$$\widetilde{\sigma} : O_{\widetilde{K'}}((X))^\times \xrightarrow{\sim} O_{\widetilde{\widetilde{\sigma}(K')}}((X))^\times$$

(c) The following diagram

$$\begin{array}{ccccc} \varprojlim_i \, (\overline{K'^i})^\times & \xleftarrow{\Delta_\omega} & \mathcal{C}(\widetilde{K'}, h) & \xrightarrow{\nabla_{\pi',h}} & \overline{\mathbb{F}}_{q_{K'}}((X))^\times \\ \Big\downarrow \widetilde{\sigma} & & \Big\downarrow \widetilde{\sigma} & & \Big\downarrow \widetilde{\sigma}^* \\ \varprojlim_i \, (\overline{\widetilde{\sigma}(K')^i})^\times & \xleftarrow{\Delta_{\widetilde{\sigma}\omega}} & \mathcal{C}(\widetilde{\widetilde{\sigma}(K')}, \widetilde{\sigma}h) & \xrightarrow{\nabla_{\widetilde{\sigma}(\pi'),\widetilde{\sigma}h}} & \overline{\mathbb{F}}_{q_{\widetilde{\sigma}(K')}}((X))^\times \end{array}$$

is commutative for any choice of $\boldsymbol{\omega} \in \Omega_h$.
(d) Suppose that K' is a compatible extension over K with respect to the Lubin-Tate splitting (K, ϕ). Then

$$N^{Coleman}_{\phi,K'/K} : \overline{\kappa}_{K'}[[X]]^\times \to \overline{\kappa}_K[[X]]^\times$$

satisfies

$$\widetilde{\sigma}^* N^{Coleman}_{\phi,K'/K}(\xi') = N^{Coleman}_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}^*(\xi'))$$

for every $\xi' \in \overline{\kappa}_{K'}[[X]]^\times$.

Proof. (a) Recall that, there exists a unique multiplicative homomorphism

$$\mathcal{N}_h : O_{\widetilde{K'}}((X))^\times \to O_{\widetilde{K'}}((X))^\times$$

satisfying

$$\mathcal{N}_h g \circ h = \prod_{\omega \in W_h^1} g(X[\underset{h}{+}]\omega)$$

for $g \in O_{\widetilde{K'}}((X))^\times$. Now, direct computation

$$\begin{aligned}
\widetilde{\sigma}(\mathcal{N}_h g \circ h) &= \prod_{\omega \in W_h^1} \widetilde{\sigma}(g(X[\underset{h}{+}]\omega)) \\
&= \prod_{\omega \in W_h^1} \widetilde{\sigma}(g(F_h(X,\omega))) \\
&= \prod_{\omega \in W_h^1} \widetilde{\sigma} g(F_{\widetilde{\sigma}h}(X, \widetilde{\sigma}\omega)) \\
&= \prod_{\delta \in W_{\widetilde{\sigma}h}} \widetilde{\sigma} g(X[\underset{\widetilde{\sigma}h}{+}]\delta) \\
&= \mathcal{N}_{\widetilde{\sigma}h} \widetilde{\sigma} g \circ \widetilde{\sigma} h
\end{aligned}$$

yields $\widetilde{\sigma}(\mathcal{N}_h g) \circ \widetilde{\sigma} h = \mathcal{N}_{\widetilde{\sigma}h} \widetilde{\sigma} g \circ \widetilde{\sigma} h$ for $g \in O_{\widetilde{K'}}((X))^\times$. The fact that $\widetilde{\sigma} g \mapsto \widetilde{\sigma}(\mathcal{N}_h g)$ for $g \in O_{\widetilde{K'}}((X))^\times$ is an endomorphism on $O_{\widetilde{\sigma(K')}}((X))^\times$ now proves the commutativity of the square.

(b) Directly follows from part (a) and by the definition of $\mathcal{C}(\widetilde{K'}, h)$.

(c) The commutativity of the right-hand square follows from the commutativity of eq. no. (4.1), and the commutativity of the left-hand square follows from the definition of the isomorphism $\Delta_\omega : \mathcal{C}(\widetilde{K'}, h) \xrightarrow{\sim} \varprojlim_i (\overline{K'^i})^\times$ for any choice of $\boldsymbol{\omega} \in \Omega_h$.

(d) let $g_{\xi'} \in \mathcal{C}^\circ(\widetilde{K'}, f_{\phi',K'})$ be the Coleman power series lifting $\xi' \in \overline{\kappa}_{K'}[[X]]^{\times \cdot}$. Clearly

$$N^{Coleman}_{\phi,K'/K}(\xi') = \nabla_{\pi_K, f_{\phi,K}} \circ \Delta_{\boldsymbol{\omega}}^{-1}\{u_n\}_{1 \leq n \in \mathbb{Z}},$$

where $\{u_n\}_{1\leq n\in\mathbb{Z}} \in \varprojlim_i U(\overline{K^i}) = \mathcal{U}(\widetilde{K})$ and

$$u_n = \prod_{0\leq \ell\leq f(K'/K)-1} \phi^\ell N_{\widetilde{K'_m}/\widetilde{K_n}}(\phi_{K'} g_{\xi'}(\pi_{K'_m}))$$

for any choice of $m \gg n$. By part (c),

$$\begin{aligned}\widetilde{\sigma}^* N^{Coleman}_{\phi,K'/K}(\xi') &= \nabla_{\widetilde{\sigma}(\pi_K), f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma K}}(\widetilde{\sigma}\circ\Delta^{-1}_{\boldsymbol{\omega}}\{u_n\}_{1\leq n\in\mathbb{Z}}) \\ &= \nabla_{\widetilde{\sigma}(\pi_K), f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma K}} \circ \Delta^{-1}_{\widetilde{\sigma}\boldsymbol{\omega}} \widetilde{\sigma}\{u_n\}_{1\leq n\in\mathbb{Z}} \\ &= N^{Coleman}_{\tilde{\sigma}\phi\tilde{\sigma}^{-1},\tilde{\sigma}(K')/\sigma(K)}(\tilde{\sigma}^*\xi')\end{aligned}$$

which completes the proof. □

If K' is a finite Galois extension over K and π' a fixed prime element in $\widetilde{K'}$ (such that $\nu_{K'}(\pi') = 1$), then the group $G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi'))$ is defined by
$G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi'))$

$$= \left\{(\delta, c) : \delta\in \text{ Gal}(\widetilde{\sigma}(K')^{nr}/\sigma(K)), c\in U(\widetilde{\sigma}(\widetilde{K'})) \text{ s.t } \frac{\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1}(c)}{c} = \frac{\delta(\widetilde{\sigma}(\pi'))}{\widetilde{\sigma}(\pi')}\right\}.$$

§5. Action of $\widetilde{\sigma}\in$ Aut (K^{sep}) on $\mathfrak{G}(K,\phi)$.

As in the previous section, let $\sigma : K \hookrightarrow K^{sep}$ be an embedding of K into K^{sep}, and let $\widetilde{\sigma} : K^{sep} \to K^{sep}$ be a fixed extension of $\sigma : K \hookrightarrow K^{sep}$ to K^{sep}.

Lemma 5.1. *Suppose that K' is a finite extension over K (inside K^{sep}), and ϕ' a Lubin-Tate splitting for K'. Then,*
(a) for $(a',\xi') \in \mathfrak{G}_d(K',\phi')$,

$$\widetilde{\sigma}^+_d(a',\xi') = (\widetilde{\sigma}(a'), \widetilde{\sigma}^*\xi') \in \mathfrak{G}_d(\widetilde{\sigma}(K'), \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1}),$$

where
$\mathfrak{G}_d(\widetilde{\sigma}(K'), \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$

$$= \left\{ (b,\eta) : \begin{array}{c} b = v\pi^\mu_{\widetilde{\sigma}(K')} \in \widetilde{\sigma}(\widehat{K'})^\times \;\; s.t\; v\in U(\widetilde{\sigma}(K')), \pi_{\widetilde{\sigma}(K')} = \widetilde{\sigma}(\pi_{K'}), \mu\in\widehat{\mathbb{Z}} \\ \eta = \eta(X) \in \overline{\kappa}_{\widetilde{\sigma}(K')}[[X]]^\times \;\; s.t\; \frac{(\widetilde{\sigma}^*\phi'^d(\widetilde{\sigma}^*)^{-1})\eta}{\eta} = \frac{\{v\}_{\widetilde{\sigma f}_{\phi',K'}}}{X} \end{array} \right\};$$

(b) suppose further that K' is a finite Galois extension over K, and π' a fixed prime element in $\widetilde{K'}$ such that $\nu_{K'}(\pi') = 1$, where $\nu_{K'}$ is the normalized valuation on K'. Then, for $(\gamma, b) \in G(K'/K, \pi')$,

$$\widetilde{\sigma}^{+}_{K'/K}(\gamma, b) = (\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b)) \in G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi')),$$

where
$G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi'))$

$$= \left\{(\delta, c) : \delta \in \ Gal(\widetilde{\sigma}(K')^{nr}/\sigma(K)), c \in U(\widetilde{\widetilde{\sigma}(K')}) \ s.t \ \frac{\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1}(c)}{c} = \frac{\delta(\widetilde{\sigma}(\pi'))}{\widetilde{\sigma}(\pi')}\right\}.$$

Proof. (a) Suppose that $(a', \xi') \in \mathfrak{G}_d(K', \phi')$ with $a' = u'\pi_{K'}^{\nu} \in \widehat{K'^{\times}}$ for some $u' \in U(K')$, $\nu \in \hat{\mathbb{Z}}$, and $\xi' \in \overline{\kappa}_{K'}[[X]]^{\times}$ such that $\frac{\phi'^d\xi'}{\xi'} = \frac{\{u'\}_{f_{\phi',K'}}}{X}$. Put $v' = \widetilde{\sigma}(u')$ and $\pi_{\widetilde{\sigma}(K')} = \widetilde{\sigma}(\pi_{K'})$. It suffices to prove that

$$\frac{(\widetilde{\sigma}^*\phi'^d(\widetilde{\sigma}^*)^{-1})\widetilde{\sigma}^*\xi'}{\widetilde{\sigma}^*\xi'} = \frac{\{v'\}_{f_{\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')}}}{X}.$$

In fact, by part (d) of Lemma 4.1, $\{v'\}_{f_{\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')}} = \widetilde{\sigma}^*\{u'\}_{f_{\phi',K'}}$ and thereby

$$\begin{aligned}\frac{(\widetilde{\sigma}^*\phi'^d(\widetilde{\sigma}^*)^{-1})\widetilde{\sigma}^*\xi'}{\widetilde{\sigma}^*\xi'} &= \widetilde{\sigma}^*\left(\frac{\phi'^d\xi'}{\xi'}\right) \\ &= \widetilde{\sigma}^*\left(\frac{\{u'\}_{f_{\phi',K'}}}{X}\right) = \frac{\widetilde{\sigma}^*\{u'\}_{f_{\phi',K'}}}{X} = \frac{\{v'\}_{f_{\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')}}}{X}\end{aligned}$$

proving that $(\widetilde{\sigma}(a'), \widetilde{\sigma}^*(\xi')) \in \mathfrak{G}_d(\widetilde{\sigma}(K'), \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$.
(b) Note that $\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1} \in \mathrm{Gal}(\widetilde{\sigma}(K')/\sigma(K)), \widetilde{\sigma}(b) \in U(\widetilde{\widetilde{\sigma}(K')})$, and

$$\frac{\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1}(\widetilde{\sigma}(b))}{\widetilde{\sigma}(b)} = \widetilde{\sigma}\left(\frac{\phi_{K'}(b)}{b}\right) = \widetilde{\sigma}\left(\frac{\gamma(\pi')}{\pi'}\right) = \frac{\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}(\widetilde{\sigma}(\pi'))}{\widetilde{\sigma}(\pi')}$$

proving that $(\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b)) \in G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi'))$. □

Following Lemma 5.1 (a), define the map

$$\widetilde{\sigma}_d^+ : \mathfrak{G}_d(K', \phi') \to \mathfrak{G}_d(K', \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$$

by

$$\widetilde{\sigma}_d^+ : (a', \xi') \mapsto (\widetilde{\sigma}a', \widetilde{\sigma}^* \xi')$$

for every $(a', \xi') \in \mathfrak{G}_d(K', \phi')$. Since

$$\widetilde{\sigma}^* \prod_{0 \le i \le \frac{d_2}{d_1} - 1} \phi'^{d_1 i}(\xi') = \prod_{0 \le i \le \frac{d_2}{d_1} - 1} (\widetilde{\sigma}^* \phi'^{d_1 i} (\widetilde{\sigma}^*)^{-1})(\widetilde{\sigma}^* \xi')$$

for positive integers d_1, d_2 with $d_1 | d_2$, the square

$$\begin{array}{ccc}
\mathfrak{G}_{d_2}(K', \phi') & \xrightarrow{\widetilde{\sigma}_{d_2}^+} & \mathfrak{G}_{d_2}(\widetilde{\sigma}K', \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1}) \\
{\scriptstyle \tau(\phi')_{d_1}^{d_2}} \downarrow & & \downarrow {\scriptstyle \tau(\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})_{d_1}^{d_2}} \\
\mathfrak{G}_{d_1}(K', \phi') & \xrightarrow{\widetilde{\sigma}_{d_1}^+} & \mathfrak{G}_{d_1}(\widetilde{\sigma}K', \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})
\end{array}$$

is commutative. Thus, passing to the projective limits, there exists a map

$$\varprojlim_d \widetilde{\sigma}_d^+ = \widetilde{\sigma}^+ : \underbrace{\varprojlim_d \mathfrak{G}_d(K', \phi')}_{\mathfrak{G}(K', \phi')} \to \underbrace{\varprojlim_d \mathfrak{G}_d(\widetilde{\sigma}K', \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})}_{\mathfrak{G}(\widetilde{\sigma}K', \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})}$$

defined by

$$\widetilde{\sigma}^+ : (a', \{\xi'_d\}) \mapsto (\widetilde{\sigma}a', \{\widetilde{\sigma}^* \xi'_d\})$$

for every $(a', \{\xi'_d\}) \in \mathfrak{G}(K', \phi')$. Furthermore, note that,

Lemma 5.2.

$$\widetilde{\sigma}^+ : \mathfrak{G}(K', \phi') \to \mathfrak{G}(\widetilde{\sigma}K', \widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$$

is an isomorphism of topological groups.

Proof. In fact, it suffices to prove that,

$$\widetilde{\sigma}_d^+ : \mathfrak{G}_d(K',\phi') \to \mathfrak{G}_d(\widetilde{\sigma}K',\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$$

is a topological isomorphism: For $(a',\xi'),(b',\psi') \in \mathfrak{G}_d(K',\phi')$, with $a' = u'\pi_{K'}^{\nu'}$, where $u' \in U(K')$ and $\nu' \in \hat{\mathbb{Z}}$,

$$\begin{aligned}
\widetilde{\sigma}_d^+((a',\xi')(b',\psi')) &= \widetilde{\sigma}_d^+(a'b',\xi'.\phi'^{\nu'}(\psi') \circ \{u'\}_{f_{\phi',K'}}) \\
&= (\widetilde{\sigma}(a'b'),\widetilde{\sigma}^*(\xi'.\phi'^{\nu'}(\psi') \circ \{u'\}_{f_{\phi',K'}})) \\
&= (\widetilde{\sigma}(a')\widetilde{\sigma}(b'),\widetilde{\sigma}^*(\xi').(\widetilde{\sigma}^*\phi'(\widetilde{\sigma}^*)^{-1})^{\nu'}(\widetilde{\sigma}^*(\psi')) \circ \widetilde{\sigma}^*\{u'\}_{f_{\phi',K'}}) \\
&= (\widetilde{\sigma}(a')\widetilde{\sigma}(b'),\widetilde{\sigma}^*(\xi').(\widetilde{\sigma}^*\phi'(\widetilde{\sigma}^*)^{-1})^{\nu'}(\widetilde{\sigma}^*(\psi')) \circ \{\widetilde{\sigma}(u')\}_{f_{\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')}})
\end{aligned}$$

by Lemma 4.1, proving that $\widetilde{\sigma}_d^+((a',\xi')(b',\psi')) = \widetilde{\sigma}_d^+(a',\xi')\widetilde{\sigma}_d^+(b',\psi')$. Moreover, the kernel and the image of the homomorphism $\widetilde{\sigma}_d^+ : \mathfrak{G}_d(K',\phi') \to \mathfrak{G}_d(\widetilde{\sigma}K',\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$ are

$$\ker\widetilde{\sigma}_d^+ =< (1_{K'},1(X)) >,$$

and

$$\widetilde{\sigma}_d^+(\mathfrak{G}_d(K',\phi')) = \mathfrak{G}_d(\widetilde{\sigma}K',\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})$$

with

$$\widetilde{\sigma}_d^+(\mathfrak{G}_d(K',\phi')^{(i,j)}) = \mathfrak{G}_d(\widetilde{\sigma}K',\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1})^{(i,j)},$$

since $\widetilde{\sigma} : \widehat{(K')}^\times \to \widetilde{\sigma}\widehat{(K')}^\times$ and $\widetilde{\sigma}^* : \overline{\kappa}_{K'}[[x]] \to \overline{\kappa}_{\widetilde{\sigma}(K')}[[X]]$ are isomorphisms, proving that $\widetilde{\sigma}_d^+$ is a topological isomorphism. □

Lemma 5.3. *Suppose that K' is a compatible extension over K with respect to the Lubin-Tate splitting (K,ϕ). The following square is commutative*

$$\begin{array}{ccc}
\mathfrak{G}(\widetilde{\sigma}(K'),\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1}) & \xrightarrow{M_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)}} & \mathfrak{G}(\sigma(K),\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}) \\
\uparrow \widetilde{\sigma}^+ & & \uparrow \widetilde{\sigma}^+ \\
\mathfrak{G}(K',\phi') & \xrightarrow{M_{\phi,K'/K}} & \mathfrak{G}(K,\phi).
\end{array}$$

That is, the 2-abelian norm map

$$M_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)} : \mathfrak{G}(\widetilde{\sigma}(K'),\widetilde{\sigma}\phi'\widetilde{\sigma}^{-1}) \to \mathfrak{G}(\sigma(K),\widetilde{\sigma}\phi\widetilde{\sigma}^{-1})$$

satisfies

$$M_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}^{+}(\alpha')) = \widetilde{\sigma}^{+}M_{\phi,K'/K}(\alpha')$$

for every $\alpha' \in \mathfrak{G}(K',\phi')$.

Proof. Suppose that $\alpha' = (a',\xi') \in \mathfrak{G}(K',\phi')$. Then, by the definition of the metabelian norm map (look at eq. no. (2.3) and (2.4)),

$$M_{\phi,K'/K}(\alpha') = (N_{K'/K}(a'), N^{Coleman}_{\phi,K'/K}(\xi')).$$

Thus
$\widetilde{\sigma}^{+}M_{\phi,K'/K}(\alpha')$

$$= (\widetilde{\sigma}N_{K'/K}(a'), \widetilde{\sigma}^{*}N^{Coleman}_{\phi,K'/K}(\xi')) = (N_{\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}(a')), \widetilde{\sigma}^{*}N^{Coleman}_{\phi,K'/K}(\xi')).$$

Hence, it suffices to prove that $\widetilde{\sigma}^{*}N^{Coleman}_{\phi,K'/K}(\xi') = N^{Coleman}_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}^{*}(\xi'))$ since, then
$\widetilde{\sigma}^{+}M_{\phi,K'/K}(\alpha')$

$$= (N_{\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}(\alpha')), N^{Coleman}_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}^{*}(\xi'))) = M_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\widetilde{\sigma}(K')/\sigma(K)}(\widetilde{\sigma}^{+}(\alpha')).$$

Now, the proof follows from part (d) of Lemma 4.3 □

If K' is finite Galois over K, following Lemma 5.1 (b), define the map

$$\widetilde{\sigma}^{+}_{K'/K} : G(K'/K,\pi') \to G(\widetilde{\sigma}(K')/\sigma(K),\widetilde{\sigma}(\pi'))$$

by

$$\widetilde{\sigma}^{+}_{K'/K} : (\gamma,b) \mapsto (\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1},\widetilde{\sigma}(b))$$

for every $(\gamma,b) \in G(K'/K,\pi')$.

Lemma 5.4.

$$\widetilde{\sigma}^+_{K'/K} : G(K'/K, \pi') \to G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi'))$$

is an isomorphism.

Proof. For $(\gamma_1, b_1), (\gamma_2, b_2) \in G(K'/K, \pi')$,

$$\begin{aligned}
\widetilde{\sigma}^+_{K'/K}((\gamma_1, b_1)(\gamma_2, b_2)) &= \widetilde{\sigma}^+_{K'/K}(\gamma_1\gamma_2, b_1\gamma_1(b_2)) \\
&= (\widetilde{\sigma}\gamma_1\gamma_2\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b_1\gamma_1(b_2))) \\
&= (\widetilde{\sigma}\gamma_1\widetilde{\sigma}^{-1}\widetilde{\sigma}\gamma_2\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b_1)\widetilde{\sigma}\gamma_1\widetilde{\sigma}^{-1}(\widetilde{\sigma}(b_2))) \\
&= (\widetilde{\sigma}\gamma_1\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b_1))(\widetilde{\sigma}\gamma_2\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b_2)) \\
&= \widetilde{\sigma}^+_{K'/K}(\gamma_1, b_1)\widetilde{\sigma}^+_{K'/K}(\gamma_2, b_2)
\end{aligned}$$

by the definition of the law of composition on $G(K'/K, \pi')$ (cf. eq. no. (3.1)). Moreover, the kernel and the image of the homomorphism $\widetilde{\sigma}^+_{K'/K}$ are

$$\ker\widetilde{\sigma}^+_{K'/K} = < (\mathrm{id}_{K'}, 1_{K'}) >,$$

and

$$\widetilde{\sigma}^+_{K'/K}(G(K'/K, \pi')) = G(\widetilde{\sigma}(K')/\sigma(K), \widetilde{\sigma}(\pi'))$$

which completes the proof. □

Suppose that $K \subseteq K' \subseteq K''$ is a tower of finite Galois extensions over K, and π'' a prime element in $\widetilde{K''}$ such that $\widetilde{N}_{K''/K'}(\pi'') = \pi'$. Observe that, for $b'' \in U(\widetilde{K''})$ and for $d = [K'' \cap K'^{nr} : K']$,

$$\begin{aligned}
\widetilde{\sigma} \prod_{0 \leq i \lneq d} \phi^i_{K'}(\widetilde{N}_{K''/K'}(b'')) &= \prod_{0 \leq i \lneq d} \widetilde{\sigma}\phi^i_{K'}\widetilde{\sigma}^{-1}(\widetilde{\sigma}\widetilde{N}_{K''/K'}(b'')) \\
&= \prod_{0 \leq i \lneq d} (\widetilde{\sigma}\phi_{K'}\widetilde{\sigma}^{-1})^i(\widetilde{N}_{\widetilde{\sigma}(K'')/\widetilde{\sigma}(K')}(\widetilde{\sigma}(b'')))
\end{aligned}$$

and $d = [\widetilde{\sigma}(K'') \cap \widetilde{\sigma}(K')^{nr} : \widetilde{\sigma}(K')]$. Thus, the following square

$$\begin{array}{ccc} G(K''/K,\pi'') & \xrightarrow{\mathbf{r}_{K''/K'}} & G(K'/K,\pi') \\ \Big\downarrow \widetilde{\sigma}^{+}_{K''/K} & & \Big\downarrow \widetilde{\sigma}^{+}_{K'/K} \\ G(\widetilde{\sigma}(K'')/\sigma(K),\widetilde{\sigma}(\pi'')) & \xrightarrow{\mathbf{r}_{\widetilde{\sigma(K'')}/\widetilde{\sigma(K')}}} & G(\widetilde{\sigma}(K')/\sigma(K),\sigma(\pi')) \end{array}$$

is commutative.

§**6. Statement of the Galois conjugation law.**

However in [4], Koch and de Shalit did neither discuss the behaviour of the 2-abelian local Artin map under Galois conjugation, nor the "transfer" law (Verlagerung) of the 2-abelian local Artin map. The aim of this work (and its continuation) is to complete [4], and present the Galois conjugation law of the 2-abelian local Artin map (resp. the "transfer" law of the 2-abelian local Artin map). We postpone the "transfer" law of the 2-abelian local Artin map to another discussion as explained before. The main theorem of this paper is the following

Theorem A (Galois conjugation). *Let $\sigma : K \hookrightarrow K^{sep}$ be any embedding of K into K^{sep}, and $\widetilde{\sigma} \in Aut(K^{sep})$ be a fixed extension of the embedding $\sigma : K \hookrightarrow K^{sep}$ to an automorphism of K^{sep}. Then,*

$$(\widetilde{\sigma}^{+}(\alpha),\sigma(K))_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \widetilde{\sigma}|_{K^{(ab)^2}}(\alpha,K)_{\phi}\widetilde{\sigma}^{-1}|_{\tilde{\sigma}(K^{(ab)^2})}$$

for every $\alpha \in \mathfrak{G}(K,\phi)$.

In particular, if $\sigma \in \text{Aut }(K)$, then

$$(\widetilde{\sigma}^{+}(\alpha),K)_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}} = \widetilde{\sigma}|_{K^{(ab)^2}}(\alpha,K)_{\phi}\widetilde{\sigma}^{-1}|_{K^{(ab)^2}}$$

for every $\alpha \in \mathfrak{G}(K,\phi)$. Moreover, if $\sigma = \text{id}_K$ and $\widetilde{\sigma}\phi\widetilde{\sigma}^{-1} = \phi$, then necessarily $\widetilde{\sigma} = \phi^n$ for some $n \in \mathbb{Z}$, since the normalizer of $<\phi>$ in Gal (K^{sep}/K) is $<\phi>$. If this is the case, then

$$((\phi^n)^{+}(\alpha),K)_{\phi} = \phi^n|_{K^{(ab)^2}}(\alpha,K)_{\phi}\phi^{-n}|_{K^{(ab)^2}}$$

for every $\alpha \in \mathfrak{G}(K,\phi)$. These special cases of Theorem A will turn out to be useful in a future investigation on non-abelian local class field theory.

§7. Proof of Theorem A.

The following proposition will be fundamental in the proof of Theorem *A*.

Proposition 7.1. *Suppose that L/K is a finite Galois extension, $\phi_L \in Gal(L^{nr}/L)$ the Frobenius automorphism over L, and π a fixed prime element in $\tilde{L}$ such that $\nu_L(\pi) = 1$, where ν_L is the normalized valuation on L. Fix a norm-compatible squence of prime elements*

$$\boldsymbol{\omega} = \{\omega_i \in \overline{L^i} : N_{\overline{L^1}/\tilde{L}}(\omega_1) = \pi\}$$

in the tower $\{\overline{L^i}\}_{1 \leq i \in \mathbb{Z}}$, where L^i denotes the abelian extension over L which is class field to $U^i(L)$ and $\overline{L^i}$ its completion. Then,
(a) $\widetilde{\sigma}(L^i) = \widetilde{\sigma}(L)^i$ for every $1 \leq i \in \mathbb{Z}$, and

$$\widetilde{\sigma}\boldsymbol{\omega} = \{\widetilde{\sigma}(\omega_i) \in \widetilde{\widetilde{\sigma}(L)}^i : \widetilde{N}_{\widetilde{\sigma}(L)^1/\widetilde{\sigma}(L)}(\widetilde{\sigma}(\omega_1)) = \widetilde{\sigma}(\pi)\}$$

is a norm-compatible sequence of prime elements in $\widetilde{\widetilde{\sigma}(L)}^i$ for $1 \leq i \in \mathbb{Z}$.
(b) For the norm-compatible sequence $\widetilde{\sigma}\boldsymbol{\omega}$, the isomorphism

$$\iota_{\widetilde{\sigma}\boldsymbol{\omega}} : G(\widetilde{\sigma}(L)/\sigma(K), \widetilde{\sigma}(\pi); \widetilde{\sigma}\phi_L\widetilde{\sigma}^{-1}) \xrightarrow{\sim} Gal(\widetilde{\sigma}(L)^{ab}/\sigma(K))$$

defined as in Proposition 3.1 satisfies

$$\iota_{\widetilde{\sigma}\boldsymbol{\omega}}(\widetilde{\sigma}^+_{L/K}(\gamma, b)) = \widetilde{\sigma}\iota_{\boldsymbol{\omega}}(\gamma, b)\widetilde{\sigma}^{-1},$$

for every $(\gamma, b) \in G(L/K, \pi; \phi_L)$.

Proof. We leave the proof of part (a) to the reader, and we will directly prove part (b). For $(\gamma, b) \in G(L/K, \pi)$,

$$\iota_{\widetilde{\sigma}\boldsymbol{\omega}}(\widetilde{\sigma}^+_{L/K}(\gamma, b)) = \iota_{\widetilde{\sigma}\boldsymbol{\omega}}(\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b))$$

is defined by the following two conditions:

$$\iota_{\widetilde{\sigma}\boldsymbol{\omega}}(\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}, \widetilde{\sigma}(b))|_{L^{nr}} = \widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}$$

and

$$\begin{aligned}\iota_{\widetilde{\sigma}\boldsymbol{\omega}}(\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1},\widetilde{\sigma}(b))(\widetilde{\sigma}\omega_i) &= \widetilde{\sigma}\phi_L^{1-i}\widetilde{\sigma}^{-1}[\widetilde{\sigma}b]_{\widetilde{\sigma}h,(\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1})\widetilde{\sigma}h}(\widetilde{\sigma}\omega_i)\\ &= \widetilde{\sigma}\phi_L^{1-i}[b]_{h,\gamma h}(\omega_i)\\ &= \widetilde{\sigma}\iota_{\boldsymbol{\omega}}(\gamma,b)(\omega_i)\\ &= \widetilde{\sigma}\iota_{\boldsymbol{\omega}}(\gamma,b)\widetilde{\sigma}^{-1}(\widetilde{\sigma}\omega_i)\end{aligned}$$

for $1 \le i \in \mathbb{Z}$, proving the assertion. □

Note that, for $\alpha_d = (a_d, \xi_d) \in \mathfrak{G}_d(K,\phi)$, by part (a) of Lemma 5.1, $\widetilde{\sigma}_d^+(\alpha_d) = (\sigma(a_d), \widetilde{\sigma}^*(\xi_d))$, and by Proposition 3.3

$$\iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},d}(\widetilde{\sigma}_d^+(\alpha_d)) = \iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},d}(\sigma(a_d),\widetilde{\sigma}^*(\xi_d))$$

$$\begin{aligned}&= \varprojlim_i \ \iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma(K)_i\sigma(K)_d^{nr}/\sigma(K)}\big((\sigma(a_d)^{-1},\sigma(K))|_{\sigma(K)_i,\sigma(K)^{nr}},\\ &\qquad \widetilde{\sigma}\phi^{1-i}\widetilde{\sigma}^{-1}\nabla^{-1}_{\sigma(\pi_K),f_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma K}}(\widetilde{\sigma}^*(\xi_d))(\pi_{\sigma(K)^i})\big)\\ &= \varprojlim_i \ \iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma(K)_i\sigma(K)_d^{nr}/\sigma(K)}\Big(\widetilde{\sigma}(a_d^{-1},K)\widetilde{\sigma}^{-1}|_{\widetilde{\sigma}(K_iK^{nr})},\widetilde{\sigma}(\phi^{1-i}\nabla^{-1}_{\pi_K,f_{\phi,K}}(\xi_d)(\pi_{K^i}))\Big)\end{aligned}$$

by the Galois conjugation law of abelian local class field theory, by part (c) of Lemma 4.3, and by parts (a) and (b) of Lemma 4.1. Thus,

$$\iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},d}(\widetilde{\sigma}_d^+(\alpha_d)) =$$

$$\varprojlim_i \ \iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},\sigma(K)_i\sigma(K)_d^{nr}/\sigma(K)}\widetilde{\sigma}^+_{\sigma(K)_i\sigma(K)_d^{nr}/\sigma(K)}((a_d^{-1},K)|_{K_iK^{nr}},\phi^{1-i}\nabla^{-1}_{\pi_K,f_{\phi,K}}(\xi_d)(\pi_{K^i}))$$

by part (b) of Lemma 5.1. Now, applying Proposition 7.1 for each $1 \le i \in \mathbb{Z}$,

$$\iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},d}(\widetilde{\sigma}_d^+(\alpha_d))$$

$$\begin{aligned}&= \varprojlim_i \ \widetilde{\sigma}|_{K_iK_d^{nr}}\iota_{\phi,K_iK_d^{nr}/K}\Big((a_d^{-1},K)|_{K_i,K^{nr}},\phi^{1-i}\nabla^{-1}_{\pi_K,f_{\phi,K}}(\xi_d)(\pi_{K^i})\Big)\widetilde{\sigma}^{-1}|_{\tilde{\sigma}(K_iK_d^{nr})}\\ &= \widetilde{\sigma}|_{K_\infty K_d^{nr}}\iota_{\phi,d}(\alpha_d)\widetilde{\sigma}^{-1}|_{\tilde{\sigma}(K_\infty K_d^{nr})}\end{aligned}$$

for every $1 \leq d \in \mathbb{Z}$. Now, passing to the projective limits (with respect to eq. no. (3.4)),

$$\begin{aligned} \iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}}(\widetilde{\sigma}^{+}(\alpha)) &= \varprojlim_{d} \iota_{\widetilde{\sigma}\phi\widetilde{\sigma}^{-1},d}(\widetilde{\sigma}_d^{+}(\alpha_d)) \\ &= \varprojlim_{d} \widetilde{\sigma}|_{K_\infty K_d^{nr}} \iota_{\phi,d}(\alpha_d)\widetilde{\sigma}^{-1}|_{\tilde{\sigma}(K_\infty K_d^{nr})} \\ &= \widetilde{\sigma}|_{K^{(ab)2}} \iota_{\phi}(\alpha)\widetilde{\sigma}^{-1}|_{\tilde{\sigma}(K^{(ab)2})} \end{aligned}$$

for every $\alpha = \{\alpha_d\} \in \mathfrak{G}(K,\phi)$, which is the Galois conjugation law, since $\iota_\phi(\alpha) = (\alpha, K)_\phi$ for every $\alpha \in \mathfrak{G}(K,\phi)$.

References

[1] K. I. Ikeda, *A note on totally-ramified, n-abelian extensions of a local field* preprint.

[2] K. I. Ikeda and M. G. Ikeda, *Two lemmas on formal power series,* Tr. J. of Math. (to appear).

[3] K. Iwasawa, *Local Class Field Theory,* Oxford University Press, Oxford-New York, 1986.

[4] H. Koch and E. de Shalit, *Metabelian local class field theory,* J. reine angew. Math, **478** (1996), 85-106.

[5] E. de Shalit, *Iwasawa Theory of Elliptic Curves with Complex Multiplication.* Perspectives in Math., vol. 3, Academic Press, Boston, 1987.

Kazım İlhan İKEDA
TÜBİTAK-Feza Gürsey Institute,
P.O. Box 6, 81220 Çengelköy,
Istanbul/TURKEY
E-mail address: ilhan@gursey.gov.tr

Received 10.06.1999