Алгебра и анализ
Том 20 (2008), №4

GENERALIZED FESENKO RECIPROCITY MAP

© K. I. IKEDA, E. SERBEST

The paper is a natural continuation and generalization of the works of Fesenko and of the authors. Fesenko's theory is carried over to infinite APF-Galois extensions L over a local field K with a finite residue-class field κ_K of $q = p^f$ elements, satisfying $\boldsymbol{\mu}_p(K^{\text{sep}}) \subset K$ and $K \subset L \subset K_{\varphi^d}$, where the residue-class degree $[\kappa_L : \kappa_K]$ is equal to d. More precisely, for such extensions L/K and a fixed Lubin–Tate splitting φ over K, a 1-cocycle

$$\boldsymbol{\Phi}_{L/K}^{(\varphi)} : \text{Gal}(L/K) \to K^\times / N_{L_0/K} L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^\diamond / Y_{L/L_0}$$

where $L_0 = L \cap K^{nr}$, is constructed, and its functorial and ramification-theoretic properties are studied. The case of $d = 1$ recovers the theory of Fesenko.

Let K be a local field (that is, a complete discrete valuation field) with a finite residue-class field κ_K of $q = p^f$ elements. Assume that $\boldsymbol{\mu}_p(K^{\text{sep}}) \subset K$. We fix a Lubin–Tate splitting φ over K (see [10]). In [1, 2, 3], Fesenko introduced a very general non-Abelian local reciprocity map

$$\Phi_{L/K}^{(\varphi)} : \text{Gal}(L/K) \to U_{\widetilde{\mathbb{X}}(L/K)}^\diamond / Y_{L/K}$$

defined for any totally ramified infinite APF-Galois extension L/K satisfying $K \subset L \subset K_\varphi$, which generalized the earlier non-Abelian local class field theories of Koch–de Shalit [10] and Gurevich [7]. In [8], we studied the basic functorial and ramification-theoretic properties of the reciprocity map of Fesenko.

In this paper, which is a natural continuation and generalization of [1, 2, 3] and [8], we extend the theory of Fesenko to infinite APF-Galois extensions L/K satisfying $K \subset L \subset K_{\varphi^d}$, where d is the residue-class degree $[\kappa_L : \kappa_K]$. More precisely, for such extensions L/K, we construct a 1-cocycle,

$$\boldsymbol{\Phi}_{L/K}^{(\varphi)} : \text{Gal}(L/K) \to K^\times / N_{L_0/K} L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^\diamond / Y_{L/L_0},$$

where $L_0 = L \cap K^{nr}$, and study its functorial and ramification-theoretic properties. Note that the case where $d = 1$ recovers the theory of Fesenko.

Key words: local fields, higher-ramification theory, APF-extensions, Fontaine-Wintenberger field of norms, Fesenko reciprocity map, generalized Fesenko reciprocity map, non-abelian local class field theory.

The organization of this paper is as follows. In the first section, we briefly review the necessary background material from Fontaine–Wintenberger theory of fields of norms. In the second section, we introduce the generalized Fesenko reciprocity map $\mathbf{\Phi}^{(\varphi)}_{L/K}$ of an extension L/K that is an infinite APF-Galois extension satisfying $K \subset L \subset K_{\varphi^d}$, where the residue-class degree $[\kappa_L : \kappa_K]$ is equal to d, and study its functorial and ramification-theoretic properties.

The material and results of this paper play a fundamental role in our construction of non-Abelian local class field theory [9], which generalizes also the Laubie theory [11].

Notation. Throughout this paper, K will denote a local field (a complete discrete valuation field) with finite residue field $O_K/\mathfrak{p}_K =: \kappa_K$ of $q_K = q = p^f$ elements, where p is a prime number; here O_K denotes the ring of integers in K with a unique maximal ideal $\mathfrak{p}_K$. Let $\boldsymbol{\nu}_K$ denote the corresponding normalized valuation on K (normalized by $\boldsymbol{\nu}_K(K^\times) = \mathbb{Z}$), and let $\widetilde{\boldsymbol{\nu}}$ be the (unique) extension of $\boldsymbol{\nu}_K$ to a fixed separable closure K^{sep} of K. For any sub-extension L/K of K^{sep}/K, the normalized form of the valuation $\widetilde{\boldsymbol{\nu}}\,|_L$ on L will be denoted by $\boldsymbol{\nu}_L$. As usual, we let K^{nr} denote the maximal unramified extension in K^{sep}, and $\widetilde{K}$ denotes the completion of K^{nr} with respect to $\boldsymbol{\nu}_{K^{nr}}$. We fix a *Lubin–Tate splitting* $\varphi_K = \varphi$ *over* K. The fixed field of the Lubin–Tate splitting φ is denoted by K_φ. Finally, let $(\pi_E)_{K\subset E\subset K_\varphi}$ be the canonical sequence of norm-compatible prime elements in finite subextensions E/K in K_φ/K. This determines a unique *Lubin–Tate labeling over* K (see Subsection 0.2 in [10]).

Acknowledgements. The first author (K. I. I) is grateful to Institut de Mathématiques, "Théorie des Groupes, Représentations et Applications", Université Paris 7, Jussieu, Paris, France, and to the School of Pure Mathematics of the Tata Institute of Fundamental Research, Mumbai, India, where some parts of this work were completed, for their hospitality and support.

The second named author (E. S.) is grateful to TÜBITAK for a fellowship, and to the School of Mathematics of the University of Nottingham, where some parts of this work were iniciated, for hospitality and support.

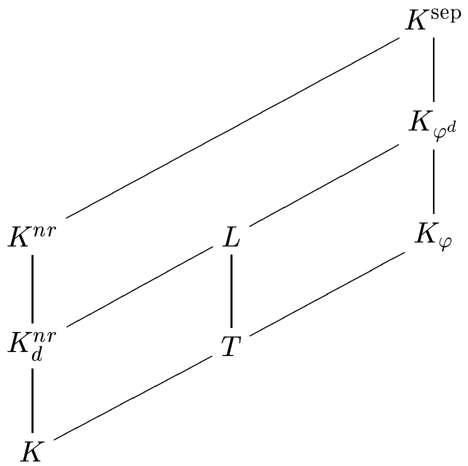
Both of the authors would like to thank I. B. Fesenko for his interest and encouragement at all stages of this work.

§1. Preliminaries on Fontaine–Wintenberger field of norms

For a brief review of APF-extensions and Fontaine–Wintenberger field of norms, we refer the reader to [8], and for detailed proofs to [5, 6, 12].

Let L/K be an infinite, Galois, arithmetically profinite (in short APF) extension such that the residue-class degree $[\kappa_L : \kappa_K]$ is equal to d and $K \subset$

$L \subset K_{\varphi^d}$; in the terminology of Koch–de Shalit in [10] and Laubie in [11], L is *compatible with* (T, φ), where T denotes the intersection field $L \cap K_\varphi$. Note that, in general, T/K is *not* a normal extension! We denote $L_0^{(K)} = L \cap K^{nr} = K_d^{nr}$. If no confusion is possible, we denote $L_0^{(K)}$ simply by L_0. So, we have the following diagram:



Remark 1.1. Note that $\varphi' = \varphi^d$ is a Lubin–Tate splitting over $L_0 = K_d^{nr}$. Therefore, by Proposition 1.2.3 in [12] or by Lemma 3.3 in [8], L/L_0 is an infinite totally ramified APF-Galois extension satisfying $L_0 \subseteq L \subseteq (L_0)_{\varphi'}$. Thus, the Fesenko theory developed in [1, 2, 3] and [8] works for the extension L/L_0.

Since L/T is an unramified extension, the following statement is true.

Lemma 1.2. *The field of norms* $\mathbb{X}(L/L_0)$ *is an unramified extension of the field of norms* $\mathbb{X}(T/K)$.

Proof. In fact, there exists a natural isomorphism $\mathbb{X}(L/L_0) \xrightarrow{\sim} \mathbb{X}(L/K)$ that identifies $\mathbb{X}(L/L_0)$ and $\mathbb{X}(L/K)$ (see Subsection (5.6) of Chapter III in [4]). Now, $\mathbb{X}(L/K)$ is a Galois extension of $\mathbb{X}(T/K)$ with the corresponding Galois group isomorphic to $\mathrm{Gal}(L/T)$ (see [8] and [12]). Since $\kappa_{\mathbb{X}(L/K)} \simeq \kappa_L$ and $\kappa_{\mathbb{X}(T/K)} \simeq \kappa_T$, it follows that

$$[\kappa_{\mathbb{X}(L/K)} : \kappa_{\mathbb{X}(T/K)}] = [\mathbb{X}(L/K) : \mathbb{X}(T/K)],$$

because L/T is an unramified extension, which proves that $\mathbb{X}(L/K)$ is an unramified extension of $\mathbb{X}(T/K)$. $\square$

Now, since the Lubin–Tate splitting φ over K is fixed, the element $\Pi_{\varphi;T/K} = (\pi_E)_{K \subset E \subset T} \in \mathbb{X}(T/K)$ is a canonical prime element of the local field $\mathbb{X}(T/K)$.

Thus, by Lemma 1.2, $\Pi_{\varphi;T/K}$ is a prime element of $\mathbb{X}(L/L_0)$ as well. Moreover, the following is true.

Lemma 1.3.

$$\Pi_{\varphi;T/K} = \Pi_{\varphi';L/L_0}. \tag{1.1}$$

Proof. Indeed, for the Lubin–Tate splitting $\varphi' = \varphi^d$ over $L_0 = K_d^{nr}$, there exists a unique element $(\pi_{L_0E})_{L_0 \subset L_0E \subset L_0T = L} \in \mathbb{X}(L/L_0)$. Since L_0E/E is an unramified extension, it follows that $\pi_{L_0E} = \pi_E$ for each $K \subset E \subset T$. Thus, (1.1) is fulfilled. □

The completion $\widetilde{\mathbb{X}}(L/K)$ of the maximal unramified extension $\mathbb{X}(L/K)^{nr}$ of the field of norms $\mathbb{X}(L/K)$ is identified with the field of norms $\mathbb{X}(\widetilde{L}/\widetilde{K}) = \mathbb{X}(\widetilde{L}/\widetilde{L_0})$.

§2. Generalized Fesenko reciprocity map

The main references for this section are [1, 2, 3] and [8].

Fix a Lubin–Tate splitting $\varphi_K = \varphi$ over K. Our aim in this section is to generalize the reciprocity map $\Phi_{M/K}^{(\varphi)}$ of Fesenko, see [1, 2, 3] and [8], defined for infinite totally ramified APF-Galois extensions M/K satisfying $K \subset M \subset K_\varphi$, to infinite APF-Galois extensions L/K with residue-class degree $[\kappa_L : \kappa_K] = d$ and satisfying $K \subset L \subset K_{\varphi^d}$. In what follows we shall keep the notation introduced in [8] and in the preceding section.

We recall that, for the extension M/K as above, the diamond subgroup $U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}$ of the group $U_{\widetilde{\mathbb{X}}(M/K)}$ of units in the ring of integers of $\widetilde{\mathbb{X}}(M/K)$ is defined by

$$U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)} = \mathrm{Pr}^{-1}_{\widetilde{K}}(U_K),$$

where $\mathrm{Pr}_{\widetilde{K}} : U_{\widetilde{\mathbb{X}}(M/K)} \to U_{\widetilde{K}}$ denotes the projection map on the $\widetilde{K}$-coordinate of $U_{\widetilde{\mathbb{X}}(M/K)}$. More generally, for a given infinite APF-Galois extension L/K with residue-class degree $[\kappa_L : \kappa_K] = d$ and satisfying $K \subset L \subset K_{\varphi^d}$, the diamond subgroup $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ of the group $U_{\widetilde{\mathbb{X}}(L/K)}$ of units in the ring of integers of $\widetilde{\mathbb{X}}(L/K) = \widetilde{\mathbb{X}}(L/L_0)$ can be defined naturally as follows.

Definition 2.1. $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ is the subgroup of the group $U_{\widetilde{\mathbb{X}}(L/K)}$ of units in the ring of integers of the local field $\widetilde{\mathbb{X}}(L/K)$ whose $\widetilde{K} = \widetilde{L}_0$-coordinate belongs to U_{L_0}. That is,

$$U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} = U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}.$$

In Fesenko's theory, which was described in [1, 2, 3] and in detail in §5 of [8], an arrow $\phi_{M/K}^{(\varphi)}$ was defined for the extensions M/K, where M/K is a totally ramified APF-Galois extension satisfying $K \subset M \subset K_\varphi$. Now, as a first step, we generalize this arrow to infinite APF-Galois extensions L of K having residue-class degree d and satisfying $K_d^{nr} \subset L \subset K_{\varphi^d}$, and construct a *generalized arrow* $\boldsymbol{\phi}_{L/K}^{(\varphi)}$ *for such extensions* L/K as follows. There exists an isomorphism

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L_0/K) \times \mathrm{Gal}(L/L_0) \tag{2.1}$$

defined by

$$\sigma \mapsto (\sigma \mid_{L_0}, \varphi^{-m}\sigma) \tag{2.2}$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $\sigma \mid_{L_0} = \varphi^m$ for some $0 \leq m \in \mathbb{Z}$.

Remark 2.2. (i) Let M/K be a Galois subextension of L/K. Let $M_0 = M \cap K^{nr}$. Then, the square

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\sim} & \mathrm{Gal}(L_0/K) \times \mathrm{Gal}(L/L_0) \\ \downarrow{\scriptstyle \mathrm{res}_M} & & \vdots{\scriptstyle (\mathrm{res}_{M_0}, \mathrm{res}_M)} \\ \mathrm{Gal}(M/K) & \xrightarrow{\sim} & \mathrm{Gal}(M_0/K) \times \mathrm{Gal}(M/M_0) \end{array}$$

is commutative. Now, for $\sigma \in \mathrm{Gal}(L/K)$, we can find $0 \leq m, m' \in \mathbb{Z}$ such that $\sigma \mid_{L_0} = \varphi^m$ and $(\sigma \mid_M) \mid_{M_0} = \varphi^{m'}$. Thus, $\varphi^m \mid_{M_0} = \varphi^{m'} \mid_{M_0}$ and the identity $(\varphi^{-m}\sigma) \mid_M = \varphi^{-m'}(\sigma \mid_M)$ is satisfied.

(ii) Let F/K be a finite Galois subextension of L/K. Suppose $L_0^{(K)} = L \cap K^{nr}$ and $L_0^{(F)} = L \cap F^{nr}$. Then, the square

$$\begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\sim} & \mathrm{Gal}(L_0^{(F)}/F) \times \mathrm{Gal}(L/L_0^{(F)}) \\ \downarrow{\scriptstyle \mathrm{inc.}} & & \vdots{\scriptstyle (\mathrm{res}_{L_0^{(K)}}, \mathrm{inc.})} \\ \mathrm{Gal}(L/K) & \xrightarrow{\sim} & \mathrm{Gal}(L_0^{(K)}/K) \times \mathrm{Gal}(L/L_0^{(K)}) \end{array}$$

is commutative. Now, for any $\sigma \in \mathrm{Gal}(L/F)$, we can find $0 \leq m, m' \in \mathbb{Z}$ such that $\sigma \mid_{L_0^{(F)}} = \varphi_F^m$ and $\sigma \mid_{L_0^{(K)}} = \varphi_K^{m'}$. Thus, $\varphi_F^m \mid_{L_0^{(K)}} = \varphi_K^{m'}$ and the identity $\varphi_F^{-m}\sigma = \varphi_K^{-m'}\sigma$ is satisfied.

By Proposition 1.2.3 in [12] or by Lemma 3.3 in [8], L/L_0 is a totally ramified APF-Galois extension with $L_0 \subset L \subset (L_0)_{\varphi'}$, where $\varphi' = \varphi^d$ is a Lubin–Tate splitting over L_0 by Remark 1.1. Thus, the map

$$\boldsymbol{\phi}_{L/K}^{(\varphi)} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^\diamond / U_{\mathbb{X}(L/K)} \tag{2.3}$$

can be defined by

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma) := \left(\pi_K^m.N_{L_0/K}L_0^\times, (u_{\widetilde{E}}).U_{\mathbb{X}(L/K)}\right), \tag{2.4}$$

where $\sigma \in \mathrm{Gal}(L/K)$ is such that $\sigma\mid_{L_0} = \varphi^m$ for some $0 \le m \in \mathbb{Z}$, and $U = (u_{\widetilde{E}}) \in U_{\widetilde{\mathbb{X}}(L/L_0)}$ satisfies

$$U^{1-\varphi^d} = \Pi_{\varphi';L/L_0}^{(\varphi^{-m}\sigma)-1}, \tag{2.5}$$

where $\Pi_{\varphi';L/L_0}$ is the canonical prime element of the local field $\mathbb{X}(L/L_0)$, which is the canonical prime element $\Pi_{\varphi;T/K}$ of the local field $\mathbb{X}(T/K)$ by Lemmas 1.2 and 1.3. Thus, (2.5) can be reformulated as

$$U^{1-\varphi^d} = \Pi_{\varphi';L/L_0}^{\sigma-1}, \tag{2.6}$$

because $\Pi^{\varphi}_{\varphi;T/K} = \Pi_{\varphi;T/K}$. Moreover, the solution $U = \left(u_{\widetilde{E}}\right) \in U_{\widetilde{\mathbb{X}}(L/L_0)}$, which is unique modulo $U_{\mathbb{X}(L/K)}$, satisfies $\mathrm{Pr}_{L_0}(U) = u_{\widetilde{L}_0} \in U_{L_0}$. In fact, $\mathrm{Pr}_{L_0}(\Pi_{\varphi';L/L_0}) = \pi_K$ by Lemma 1.3, whence $\mathrm{Pr}_{L_0}(\Pi^{\sigma-1}_{\varphi';L/L_0}) = \pi_K^{\sigma-1} = 1_K$. Hence, $\mathrm{Pr}_{L_0}(U^{1-\varphi^d}) = \mathrm{Pr}_{L_0}(\Pi^{\sigma-1}_{\varphi';L/L_0}) = 1_K$ yields $u_{\widetilde{L}_0}^{1-\varphi^d} = 1_K$, so that $u_{\widetilde{L}_0} \in U_{L_0}$ because $\widetilde{L}_0 \cap (L_0)_{\varphi'} = L_0$. Now, it follows that $\mathrm{Pr}_{L_0}(U) = u_{\widetilde{L}_0} \in U_{L_0}$. Thus, $U = (u_{\widetilde{E}})$ belongs to $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, by Definition 2.1.

Remark 2.3. We can reformulate the definition of the generalized arrow

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$$

for the extension L/K as follows:

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma) = \left(\pi_K^m.N_{L_0/K}L_0^\times, \phi^{(\varphi')}_{L/L_0}(\varphi^{-m}\sigma)\right)$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $\sigma\mid_{L_0} = \varphi^m$ for some $0 \le m \in \mathbb{Z}$.

There is a natural continuous action of $\mathrm{Gal}(L/K)$ on the topological group $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$, defined by Abelian local class field theory on the first component and by formulas (5.5) and (5.7) in [8] on the second component:

$$(\overline{a}, \overline{U})^\sigma = \left(\overline{a}^{\varphi^m}, \overline{U}^{\varphi^{-m}\sigma}\right) = \left(\overline{a}, \overline{U}^{\varphi^{-m}\sigma}\right) \tag{2.7}$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $\sigma\mid_{L_0} = \varphi^m$ for some $0 \le m \in \mathbb{Z}$, and for every $a \in K^\times$ with $\overline{a} = a.N_{L_0/K}L_0^\times$ and every $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$ with $\overline{U} = U.U_{\mathbb{X}(L/K)}$. *Below we shall always view* $K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ *as a topological Gal*(L/K)*-module.*

Theorem 2.4. *Let L/K be any infinite APF-Galois subextension of K_{φ^d}/K with residue-class degree d. Then the generalized arrow*

$$\boldsymbol{\phi}_{L/K}^{(\varphi)} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$$

defined for the extension L/K is an injection, and for every $\sigma, \tau \in \mathrm{Gal}(L/K)$, the cocycle condition

$$\boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma\tau) = \boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma)\boldsymbol{\phi}_{L/K}^{(\varphi)}(\tau)^\sigma \tag{2.8}$$

is satisfied.

Proof. The injectivity of the arrow (2.3) defined by (2.4) is clear from the canonical topological isomorphism (2.2) combined with Abelian local class field theory and Theorem 5.6 of Fesenko in [8]. More precisely, let $\boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma) = (\pi_K^m, (u_{\widetilde{E}}))$, where $d \mid m$ and $(u_{\widetilde{E}}) \in U_{\mathbb{X}(L/L_0)} = U_{\mathbb{X}(L/K)}$. Since $d \mid m$, the action of σ is trivial on L_0. Since $(u_{\widetilde{E}})^{\varphi^d - 1} = (1_{\widetilde{E}}) = \Pi^{\sigma-1}_{\varphi';L/L_0}$, σ acts trivially on the prime elements of finite subextensions between L_0 and L. Thus, σ is the identity element of $\mathrm{Gal}(L/L_0)$. Now, for $\sigma, \tau \in \mathrm{Gal}(L/K)$, with $\sigma\mid_{L_0} = \varphi^m$ and $\tau\mid_{L_0} = \varphi^n$ for some $0 \le m, n \in \mathbb{Z}$, we can use the alternative definition of the generalized arrow $\boldsymbol{\phi}_{L/K}^{(\varphi)}$, introduced in Remark 2.3, to show that

$$\begin{aligned}
\boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma\tau) &= (\pi_K^{m+n}.N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi')}(\varphi^{-(m+n)}\sigma\tau)) \\
&= \left((\pi_K^m.N_{L_0/K}L_0^\times)(\pi_K^n.N_{L_0/K}L_0^\times), \phi_{L/L_0}^{(\varphi')}(\varphi^{-m}\sigma)\phi_{L/L_0}^{(\varphi')}(\varphi^{-n}\tau)^{\varphi^{-m}\sigma}\right) \\
&= \left(\pi_K^m.N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi')}(\varphi^{-m}\sigma)\right)\left(\pi_K^n.N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi')}(\varphi^{-n}\tau)^{\varphi^{-m}\sigma}\right) \\
&= \boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma)\boldsymbol{\phi}_{L/K}^{(\varphi)}(\tau)^\sigma
\end{aligned}$$

by [8, Theorem 5.6] and by the definition of the action of $\sigma \in \mathrm{Gal}(L/K)$ on $\boldsymbol{\phi}_{L/K}^{(\varphi)}(\tau) \in K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$, defined by (2.7). □

Now, we immediately arrive at the following result.

Corollary 2.5. *Let a law of composition $*$ be defined on $\mathrm{im}(\boldsymbol{\phi}_{L/K}^{(\varphi)})$ by*

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}.\overline{b}, \overline{U}.\overline{V}^{(\phi_{L/L_0}^{(\varphi')})^{-1}(\overline{U})}) \tag{2.9}$$

for $(\overline{a}, \overline{U})$, $(\overline{b}, \overline{V}) \in \mathrm{im}(\boldsymbol{\phi}_{L/K}^{(\varphi)})$, where $\overline{a} = a.N_{L_0/K}L_0^\times$ and $\overline{b} = b.N_{L_0/K}L_0^\times \in K^\times/N_{L_0/K}L_0^\times$ with $a, b \in K^\times$, and for $\overline{U} = U.U_{\mathbb{X}(L/K)}$ and $\overline{V} = V.U_{\mathbb{X}(L/K)} \in$

$U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$ *with* $U, V \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$. *Then* $\mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{L/K})$ *is a topological group under* $*$, *and the map* $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ *induces an isomorphism of topological groups*

$$\boldsymbol{\phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{im}(\boldsymbol{\phi}^{(\varphi)}_{L/K}), \tag{2.10}$$

where the topological group structure on $im(\boldsymbol{\phi}^{(\varphi)}_{L/K})$ *is defined with respect to the binary operation* $*$ *defined by* (2.9).

Recall that, for any infinite APF-Galois extension L/K and every $-1 \leq u \in \mathbb{R}$, the uth higher ramification subgroup $\mathrm{Gal}(L/K)_u$ of $\mathrm{Gal}(L/K)$ in lower numbering is defined by

$$\mathrm{Gal}(L/K)_u = \mathrm{Gal}(L/K)^{\varphi_{L/K}(u)},$$

where $-1 \leq \varphi_{L/K}(u) \in \mathbb{R}$ is the number defined by formula (3.1) in [8], and that, as usual, the $\varphi_{L/K}(u)$th higher ramification subgroup $\mathrm{Gal}(L/K)^{\varphi_{L/K}(u)}$ of $\mathrm{Gal}(L/K)$ in upper numbering is defined to be the projective limit

$$\mathrm{Gal}(L/K)^{\varphi_{L/K}(u)} = \varprojlim_{K \subseteq F \subset L} \mathrm{Gal}(F/K)^{\varphi_{L/K}(u)},$$

see [8, (2.1) and (2.2)]. Now, let E/K be a Galois subextension of L/K. Then, for any chain of field extensions $\overbrace{\underbrace{K \subseteq F}_{\text{finite Gal.}} \subseteq F'}^{\text{finite Gal.}} \subset L$, the square

$$\begin{array}{ccc} \mathrm{Gal}(F'/K)^{\varphi_{L/K}(u)} & \xrightarrow{t^{F'}_{F'\cap E}(\varphi_{L/K}(u))} & \mathrm{Gal}(F'\cap E/K)^{\varphi_{L/K}(u)} \\ {\scriptstyle t^{F'}_{F}(\varphi_{L/K}(u))}\downarrow & & \downarrow{\scriptstyle t^{F'\cap E}_{F\cap E}(\varphi_{L/K}(u))} \\ \mathrm{Gal}(F/K)^{\varphi_{L/K}(u)} & \xrightarrow{t^{F}_{F\cap E}(\varphi_{L/K}(u))} & \mathrm{Gal}(F\cap E/K)^{\varphi_{L/K}(u)} \end{array} \tag{2.11}$$

is commutative. Thus, passing to the projective limits, we see that there exists a continuous group homomorphism

$$\begin{aligned} t^L_E(\varphi_{L/K}(u)) = \varprojlim_{K \subseteq F \subset L} t^F_{F\cap E}(\varphi_{L/K}(u)) &: \mathrm{Gal}(L/K)^{\varphi_{L/K}(u)} \\ &\to \mathrm{Gal}(E/K)^{\varphi_{L/K}(u)}, \end{aligned} \tag{2.12}$$

which is essentially the restriction morphism from L to E. This morphism is a surjection, because the objects in the respective projective systems are compact and Hausdorff. Furthermore, the square

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\quad r_E^L \quad} & \mathrm{Gal}(E/K) \\ \uparrow{\scriptstyle \text{inc.}} & & \uparrow{\scriptstyle \text{inc.}} \\ \mathrm{Gal}(L/K)^{\varphi_{L/K}(u)} & \xrightarrow{t_E^L(\varphi_{L/K}(u))} & \mathrm{Gal}(E/K)^{\varphi_{L/K}(u)} \\ \uparrow{\scriptstyle \text{inc.}} & & \uparrow{\scriptstyle \text{inc.}} \\ \mathrm{Gal}(L/K)^{\varphi_{L/K}(u')} & \xrightarrow{t_E^L(\varphi_{L/K}(u'))} & \mathrm{Gal}(E/K)^{\varphi_{L/K}(u')} \end{array} \tag{2.13}$$

is commutative for every pair $u, u' \in \mathbb{R}_{\geq -1}$ satisfying $u \leq u'$. Here, the arrow $r_E^L : \mathrm{Gal}(L/K) \to \mathrm{Gal}(E/K)$ is the restriction map. Therefore, we arrive at the following result.

Lemma 2.6. *For $0 \leq u \in \mathbb{R}$, the topological isomorphism defined by* (2.1) *and* (2.2) *induces a topological isomorphism*

$$\mathrm{Gal}(L/K)_u \simeq \underbrace{\mathrm{Gal}(L_0/K)^{\varphi_{L/K}(u)}}_{\langle \mathrm{id}_{L_0} \rangle} \times \mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)} \tag{2.14}$$

defined by

$$\sigma \mapsto \left(t_{L_0}^L(\varphi_{L/K}(u))(\sigma), \varphi^{-m}\sigma \right) = (\mathrm{id}_{L_0}, \sigma) \tag{2.15}$$

for every $\sigma \in \mathrm{Gal}(L/K)_u$ with $\sigma \mid_{L_0} = t_{L_0}^L(\varphi_{L/K}(u))(\sigma) = \varphi^m \mid_{L_0}$ for some $0 \leq m \in \mathbb{Z}$ satisfying $d \mid m$.

Proof. Note that, for $0 \leq u \in \mathbb{R}$, $\mathrm{Gal}(L_0/K)^{\varphi_{L/K}(u)}$ is the trivial group $< \mathrm{id}_{L_0} >$, because L_0/K is a finite unramified extension. Thus, for $\sigma \in \mathrm{Gal}(L/K)_u$, by the commutativity of the diagram (2.13) we have

$$t_{L_0}^L(\varphi_{L/K}(u))(\sigma) = \sigma \mid_{L_0} = \mathrm{id}_{L_0},$$

and in return

$$\sigma \mapsto \left(\mathrm{id}_{L_0}, \varphi^{-m}\sigma\right) = (\mathrm{id}_{L_0}, \sigma),$$

where $\sigma \mid_{L_0} = \mathrm{id}_{L_0} = \varphi^m \mid_{L_0}$ for some $0 \leq m \in \mathbb{Z}$ satisfying $d \mid m$. Since L is fixed by φ^d, we have $\varphi^{-m}\sigma = \sigma$, so that $(\mathrm{id}_{L_0}, \varphi^{-m}\sigma) = (\mathrm{id}_{L_0}, \sigma)$. Now, the injectivity of the morphism (2.14), defined by (2.15), is clear from the commutative square (2.13) and from the injectivity of the arrow defined by (2.2). Thus, it suffices to prove that this morphism is a surjection, which follows from the triviality of $\mathrm{Gal}(L_0/K)^{\varphi_{L/K}(u)}$ for $0 \leq u \in \mathbb{R}$, and from the fact that $\mathrm{Gal}(L/K)_u = \mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)}$. □

Now, L/L_0 is an APF-Galois subextension of L/K by part (i) of Lemma 3.3 in [8]. Let $\varphi_{L/L_0} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be the Hasse–Herbrand function corresponding to the APF-extension L/L_0 defined by relation (3.1) in [8], which is piecewise-linear and continuous. So, there exists a unique number $w = w(u, L/K) \in \mathbb{R}_{\geq -1}$ depending on u, satisfying $\varphi_{L/K}(u) = \varphi_{L/L_0}(w)$, and such that

$$\mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)} = \mathrm{Gal}(L/L_0)^{\varphi_{L/L_0}(w)} = \mathrm{Gal}(L/L_0)_w.$$

Thus, Lemma 2.6 can be reformulated as follows. The topological isomorphism defined by (2.1) and (2.2) induces a topological isomorphism

$$\mathrm{Gal}(L/K)_u \simeq \langle \mathrm{id}_{L_0} \rangle \times \mathrm{Gal}(L/L_0)_{w(u,L/K)}$$

for every $0 \leq u \in \mathbb{R}$.

For each $0 \leq i \in \mathbb{R}$, we consider the ith higher unit group $U^i_{\widetilde{\mathbb{X}}(L/K)}$ of the field $\widetilde{\mathbb{X}}(L/K)$, and introduce the group

$$\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^i = U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)} \cap U^i_{\widetilde{\mathbb{X}}(L/K)}. \tag{2.16}$$

Now, the Fesenko ramification theorem, stated as Theorem 5.8 in [8], has the following generalization for the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ corresponding to the extension L/K that is an infinite APF-Galois subextension of K_{φ^d}/K with residue-class degree $[\kappa_L : \kappa_K] = d$.

Theorem 2.7 (Ramification theorem). *For $0 \leq u \in \mathbb{R}$, let $\mathrm{Gal}(L/K)_u$ denote the uth higher ramification subgroup in the lower numbering of the Galois group $\mathrm{Gal}(L/K)$ corresponding to the infinite APF-Galois subextension L/K of K_{φ^d}/K with residue-class degree $[\kappa_L : \kappa_K] = d$. Then, for $0 \leq n \in \mathbb{Z}$, we have the inclusion*

$$\begin{aligned}
&\boldsymbol{\phi}^{(\varphi)}_{L/K}\left(\mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n)} - \mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n+1)}\right)\\
&\quad\subseteq \left\langle 1_{K^\times/N_{L_0/K}L_0^\times}\right\rangle\\
&\quad\times\left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^n U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} - \left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^{n+1} U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}\right).
\end{aligned}$$

Proof. We start with the following general observation. Let $0 \leq u \in \mathbb{R}$. Let $\tau \in \mathrm{Gal}(L/K)_u = \mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)}$. Then, by the definition of the generalized arrow $\boldsymbol{\phi}^{(\varphi)}_{L/K}$ reformulated as in Remark 2.3,

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\tau) = \left(\pi_K^m . N_{L_0/K}L_0^\times, \phi^{(\varphi')}_{L/L_0}(\varphi^{-m}\tau)\right),$$

where $\tau\mid_{L_0}=\varphi^m\mid_{L_0}$ for some $0\le m\in\mathbb{Z}$ satisfying $d\mid m$, because $\tau\in\mathrm{Gal}(L/K)_u$ and $\tau\mid_{L_0}=t_{L_0}^L(\varphi_{L/K}(u))(\tau)\in\mathrm{Gal}(L_0/K)^{\varphi_{L/K}(u)}=\langle\mathrm{id}_{L_0}\rangle$. Thus,

$$\boldsymbol{\phi}_{L/K}^{(\varphi)}(\tau)=\left(1_{K^\times/N_{L_0/K}L_0^\times},\phi_{L/L_0}^{(\varphi')}(\varphi^{-m}\tau)\right),$$

as $m=dm'$ and thereby $\pi_K^{dm'}N_{L_0/K}L_0^\times=N_{L_0/K}L_0^\times=1_{K^\times/N_{L_0/K}L_0^\times}$, because $N_{L_0/K}\pi_K^{m'}=\pi_K^m$. Therefore, we have

$$\boldsymbol{\phi}_{L/K}^{(\varphi)}(\tau)=\left(1_{K^\times/N_{L_0/K}L_0^\times},\phi_{L/L_0}^{(\varphi')}(\tau)\right).$$

Indeed, $\varphi^{-m}\tau=\tau$ in $\mathrm{Gal}(L/L_0)$ because $d\mid m$ and $L\subset K_{\varphi^d}$.

Now, to prove the theorem, we put $u=\psi_{L/K}\circ\varphi_{L/L_0}(n)$ and $u'=\psi_{L/K}\circ\varphi_{L/L_0}(n+1)$. Then, for any $\tau\in\mathrm{Gal}(L/K)_u-\mathrm{Gal}(L/K)_{u'}$, the Fesenko ramification theorem (see [8, Theorem 5.8]) shows that the second coordinate of $\boldsymbol{\phi}_{L/K}^{(\varphi)}(\tau)$ satisfies

$$\phi_{L/L_0}^{(\varphi')}(\tau)\in\left(U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}\right)^nU_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)}-\left(U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}\right)^{n+1}U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)},$$

because

$$\mathrm{Gal}(L/K)_u=\mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)}=\mathrm{Gal}(L/L_0)^{\varphi_{L/L_0}(n)}=\mathrm{Gal}(L/L_0)_n$$

and likewise

$$\mathrm{Gal}(L/K)_{u'}=\mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u')}=\mathrm{Gal}(L/L_0)^{\varphi_{L/L_0}(n+1)}=\mathrm{Gal}(L/L_0)_{n+1},$$

which completes the proof. □

Now, let M/K be an infinite Galois subextension of L/K. Thus, by [8, Lemma 3.3], M is an APF-Galois extension over K. We further assume that the residue-class degree $[\kappa_M:\kappa_K]$ is equal to d' and $K\subset M\subset K_{\varphi^{d'}}$ for some $d'\mid d$. Let

$$\boldsymbol{\phi}_{M/K}^{(\varphi)}:\mathrm{Gal}(M/K)\to K^\times/N_{M_0/K}M_0^\times\times U_{\widetilde{\mathbb{X}}(M/K)}^{\diamond}/U_{\mathbb{X}(M/K)}$$

be the corresponding generalized arrow defined for the extension M/K. Here, $M_0=M\cap K^{nr}=K_{d'}^{nr}$.

Let

$$K\subset L_o=E_o\subset E_1\subset\cdots\subset E_i\subset\cdots\subset L$$

be an ascending chain satisfying $L=\bigcup_{0\le i\in\mathbb{Z}}E_i$ and $[E_{i+1}:E_i]<\infty$ for every $0\le i\in\mathbb{Z}$. Then

$$K\subset M_o=E_o\cap M\subseteq E_1\cap M\subseteq\cdots\subseteq E_i\cap M\subseteq\cdots\subset M$$

is an ascending chain of field extensions satisfying $M = \bigcup_{0\leq i\in\mathbb{Z}} (E_i \cap M)$ and also $[E_{i+1}\cap M : E_i \cap M] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Thus, we construct $\mathbb{X}(M/K)$ by the sequence $(E_i \cap M)_{0\leq i\in\mathbb{Z}}$ and $\widetilde{\mathbb{X}}(M/K)$ by the sequence $(\widetilde{E_i \cap M})_{0\leq i\in\mathbb{Z}}$. Observe that $E_i \cap M \neq E_i$ for every $0 \leq i \in \mathbb{Z}$. Furthermore, the commutative square

$$\begin{array}{ccc} \widetilde{E}_i^{\times} & \xleftarrow{\ \widetilde{N}_{E_{i'}/E_i}\ } & \widetilde{E}_{i'}^{\times} \\ {\scriptstyle \prod_{0\leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}} \Big\downarrow & & \Big\downarrow {\scriptstyle \prod_{0\leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell}\widetilde{N}_{E_{i'}/E_{i'}\cap M}} \\ \widetilde{E_i \cap M}^{\times} & \xleftarrow{\ \widetilde{N}_{E_{i'}\cap M/E_i\cap M}\ } & \widetilde{E_{i'} \cap M}^{\times} \end{array}$$

for every pair $0 \leq i, i' \in \mathbb{Z}$ satisfying $i \leq i'$, induces the group homomorphism

$$\widetilde{\mathcal{N}}_{L/M} = \varprojlim_{0\leq i\in\mathbb{Z}} \left(\prod_{0\leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell} \widetilde{N}_{E_i/E_i\cap M} \right) : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times} \tag{2.17}$$

defined by

$$\widetilde{\mathcal{N}}_{L/M}\left((\alpha_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}} \right) = \left(\prod_{0\leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell} \widetilde{N}_{E_i/E_i\cap M}(\alpha_{\widetilde{E}_i}) \right)_{0\leq i\in\mathbb{Z}}, \tag{2.18}$$

for every $(\alpha_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}} \in \widetilde{\mathbb{X}}(L/K)^{\times}$.

Remark 2.8. The group homomorphism

$$\widetilde{\mathcal{N}}_{L/M} : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times}$$

defined by (2.17) and (2.18) does *not* depend on the choice of the ascending chain

$$K \subset L_o = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

satisfying $L = \bigcup_{0\leq i\in\mathbb{Z}} E_i$ and $[E_{i+1} : E_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$.

Remark 2.9. For $0 \leq i \in \mathbb{Z}$, let $E_{i,0}^{(E_i\cap M)} = E_i \cap (E_i \cap M)^{nr}$ be the maximal unramified extension of $E_i \cap M$ inside E_i. To simplify the notation, we put $E_{i,0} = E_i \cap (E_i \cap M)^{nr}$. Then the Galois group $\mathrm{Gal}(E_{i,0}/E_i \cap M)$ is cyclic of order $f(L/M) = \frac{d}{d'}$ and is generated by $\varphi^{d'}$. Thus, for $\alpha \in E_i$,

$$N_{E_i/E_i\cap M}(\alpha) = \widetilde{N}_{E_i/E_i\cap M}(\alpha)^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}}.$$

The basic properties of this group homomorphism are listed below.

(i) If $U = (u_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in U_{\widetilde{\mathbb{X}}(L/K)}$, then $\widetilde{\mathcal{N}}_{L/M}(U) \in U_{\widetilde{\mathbb{X}}(M/K)}$.

Proof. Indeed, using the definition of the valuation $\nu_{\widetilde{\mathbb{X}}(M/K)}$ of $\widetilde{\mathbb{X}}(M/K)$ and the definition of the valuation $\nu_{\widetilde{\mathbb{X}}(L/K)}$ of $\widetilde{\mathbb{X}}(L/K)$, we obtain

$$\nu_{\widetilde{\mathbb{X}}(M/K)}\left(\widetilde{\mathcal{N}}_{L/M}(U)\right) = \nu_{\widetilde{\mathbb{X}}(M/K)}\left(\left(\prod_{0 \leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell} \widetilde{N}_{E_i/E_i \cap M}(u_{\widetilde{E}_i})\right)_{0 \leq i \in \mathbb{Z}}\right)$$
$$= \nu_{\widetilde{K}}\left(\prod_{0 \leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell}(u_{\widetilde{K}})\right) = \sum_{0 \leq \ell \lneq f(L/M)} \nu_{\widetilde{K}}\left((\varphi^{d'})^{\ell}(u_{\widetilde{K}})\right)$$
$$= \sum_{0 \leq \ell \lneq f(L/M)} \nu_{\widetilde{K}}(u_{\widetilde{K}}) = 0,$$

because $\nu_{\widetilde{K}}\left((\varphi^{d'})^{\ell}(u_{\widetilde{K}})\right) = \nu_{\widetilde{K}}(u_{\widetilde{K}})$ for $\ell = 1, \cdots, f(L/M) - 1$, and

$$\nu_{\widetilde{\mathbb{X}}(L/K)}(U) = \nu_{\widetilde{K}}(u_{\widetilde{K}}) = 0,$$

because $U \in U_{\widetilde{\mathbb{X}}(L/K)}$. □

(ii) If $U = (u_{\widetilde{E}_i})_{0 \leq i \in \mathbb{Z}} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, then $\widetilde{\mathcal{N}}_{L/M}(U) \in U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}$.

Proof. Note that $\widetilde{L}_0 = \widetilde{K}$ and $\widetilde{M_0} = \widetilde{K}$. Now, the claim follows from the observation that

$$\mathrm{Pr}_{\widetilde{K}}(U) = u_{\widetilde{K}} \in U_{L_0}$$

and

$$\mathrm{Pr}_{\widetilde{K}}\left(\widetilde{\mathcal{N}}_{L/M}(U)\right) = \prod_{0 \leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell} \widetilde{N}_{E_o/E_o \cap M}(u_{\widetilde{E}_o})$$
$$= \prod_{0 \leq \ell \lneq f(L/M)} (\varphi^{d'})^{\ell} u_{\widetilde{K}} = N_{E_o/E_o \cap M}(u_{\widetilde{K}}) \in U_{M_0}.$$

□

(iii) If $U = (u_{E_i})_{0 \leq i \in \mathbb{Z}} \in U_{\mathbb{X}(L/K)}$, then $\widetilde{\mathcal{N}}_{L/M}(U) \in U_{\mathbb{X}(M/K)}$.

Proof. This follows from the definition (2.18) of the homomorphism (2.17), combined with the fact that

$$\widetilde{N}_{E_i/E_i \cap M}(u_{E_i})^{1 + \varphi^{d'} + \cdots + \varphi^{d'(f(L/M)-1)}} = N_{E_i/E_i \cap M}(u_{E_i})$$

for every $u_{E_i} \in U_{E_i}$ and every $0 \leq i \in \mathbb{Z}$, by Remark 2.9. □

Observe that $\widetilde{N}_{E_i/E_{i-1}}\left(\alpha^{1+\varphi^{d'}+\cdots+\varphi^{d'(f-1)}}\right) = \widetilde{N}_{E_i/E_{i-1}}(\alpha)^{1+\varphi^{d'}+\cdots+\varphi^{d'(f-1)}}$ for any $\alpha \in \widetilde{E}_i$ with $1 \leq i \in \mathbb{Z}$, where $f = f(L/M)$. Thus, we can define a homomorphism

$$\langle\varphi\rangle_{L/M} : \widetilde{\mathbb{X}}(L/L_0)^\times \to \widetilde{\mathbb{X}}(L/L_0)^\times \tag{2.19}$$

by the rule

$$\langle\varphi\rangle_{L/M} : \left(\alpha_{\widetilde{E}_i}\right)_{0\leq i\in\mathbb{Z}} \mapsto \left(\alpha_{\widetilde{E}_i}^{1+\varphi^{d'}+\cdots+\varphi^{d'(f-1)}}\right)_{0\leq i\in\mathbb{Z}} \tag{2.20}$$

for every $\left(\alpha_{\widetilde{E}_i}\right)_{0\leq i\in\mathbb{Z}} \in \widetilde{\mathbb{X}}(L/L_0)^\times$. The basic properties of this group homomorphism are listed below.

(i) $\langle\varphi\rangle_{L/M}\left(U_{\widetilde{\mathbb{X}}(L/L_0)}\right) \subseteq U_{\widetilde{\mathbb{X}}(L/L_0)}$.

Proof. Indeed, the definition of the valuation $\nu_{\widetilde{\mathbb{X}}(L/L_0)}$ of $\widetilde{\mathbb{X}}(L/L_0)$ shows that

$$\begin{aligned}
\nu_{\widetilde{\mathbb{X}}(L/L_0)}\left(\langle\varphi\rangle_{L/M}(U)\right) &= \nu_{\widetilde{\mathbb{X}}(L/L_0)}\left(u_{\widetilde{E}_i}^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}}\right)_{0\leq i\in\mathbb{Z}} \\
&= \nu_{\widetilde{K}}\left(u_{\widetilde{K}}^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}}\right) = \sum_{0\leq \ell \lneq f(L/M)} \nu_{\widetilde{K}}\left((\varphi^{d'})^\ell(u_{\widetilde{K}})\right) \\
&= \sum_{0\leq \ell \lneq f(L/M)} \nu_{\widetilde{K}}(u_{\widetilde{K}}) = 0,
\end{aligned}$$

because $\nu_{\widetilde{K}}\left((\varphi^{d'})^\ell(u_{\widetilde{K}})\right) = \nu_{\widetilde{K}}(u_{\widetilde{K}})$ for $\ell = 1, \cdots, f(L/M)-1$, and

$$\nu_{\widetilde{\mathbb{X}}(L/K)}(U) = \nu_{\widetilde{K}}(u_{\widetilde{K}}) = 0,$$

because $U \in U_{\widetilde{\mathbb{X}}(L/K)}$. □

(ii) $\langle\varphi\rangle_{L/M}\left(U^\diamond_{\widetilde{\mathbb{X}}(L/L_0)}\right) \subseteq U^\diamond_{\widetilde{\mathbb{X}}(L/L_0)}$.

Proof. Note that $\widetilde{L}_0 = \widetilde{K}$. Now, the claim follows from the observation that

$$\mathrm{Pr}_{\widetilde{K}}(U) = u_{\widetilde{K}} \in U_{L_0}$$

and

$$\begin{aligned}
\mathrm{Pr}_{\widetilde{K}}\left(\langle\varphi\rangle_{L/M}(U)\right) &= u_{\widetilde{K}}^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}} \\
&= \prod_{0\leq \ell \lneq f(L/M)} (\varphi^{d'})^\ell u_{\widetilde{K}} = N_{E_o/E_o\cap M}(u_{\widetilde{K}}) \in U_{M_0} \subseteq U_{L_0}.
\end{aligned}$$

□

(iii) $\langle\varphi\rangle_{L/M}\left(U_{\mathbb{X}(L/L_0)}\right) \subseteq U_{\mathbb{X}(L/L_0)}$.

Proof. Clearly, for $U = (u_{E_i})_{0\leq i\in\mathbb{Z}} \in U_{\mathbb{X}(L/L_o)}$ we have

$$\langle\varphi\rangle_{L/M}(U) = (u_{E_i}^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}})_{0\leq i\in\mathbb{Z}} \in U_{\mathbb{X}(L/L_o)}$$

because $u_{E_i} \in U_{E_i}$ for every $0 \leq i \in \mathbb{Z}$. □

Thus, there exists a group homomorphism

$$\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M} : \widetilde{\mathbb{X}}(L/K)^\times \to \widetilde{\mathbb{X}}(M/K)^\times \tag{2.21}$$

satisfying

(i) $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left(U_{\widetilde{\mathbb{X}}(L/K)}\right) \subseteq U_{\widetilde{\mathbb{X}}(M/K)}$;

(ii) $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left(U^\diamond_{\widetilde{\mathbb{X}}(L/K)}\right) \subseteq U^\diamond_{\widetilde{\mathbb{X}}(M/K)}$;

(iii) $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}\left(U_{\mathbb{X}(L/K)}\right) \subseteq U_{\mathbb{X}(M/K)}$.

Now, we introduce the *Coleman norm map*

$$\widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M} : U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^\diamond_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)} \tag{2.22}$$

from L to M by

$$\widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}(\overline{U}) = \widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U).U_{\mathbb{X}(M/K)} \tag{2.23}$$

for every $U \in U^\diamond_{\widetilde{\mathbb{X}}(L/K)}$, where, as before, $\overline{U}$ denotes the coset $U.U_{\mathbb{X}(L/K)}$ in $U^\diamond_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$.

Lemma 2.10. *For an infinite Galois subextension M/K of L/K such that the residue-class degree $[\kappa_M : \kappa_K]$ is equal to d' and $K \subset M \subset K_{\varphi^{d'}}$ for some $d' \mid d$, the square*

$$\begin{array}{ccc}
\mathrm{Gal}(L/L_0) & \xrightarrow{\phi^{(\varphi^d)}_{L/L_0}} & U^\diamond_{\widetilde{\mathbb{X}}(L/L_0)}/U_{\mathbb{X}(L/L_0)} \\
\Big\downarrow{\scriptstyle \mathrm{res}_M} & & \Big\downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\
\mathrm{Gal}(M/M_0) & \xrightarrow{\phi^{(\varphi^{d'})}_{M/M_0}} & U^\diamond_{\widetilde{\mathbb{X}}(M/M_0)}/U_{\mathbb{X}(M/M_0)},
\end{array} \tag{2.24}$$

where the right-vertical arrow is the Coleman norm map $\widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}$ from L to M defined by (2.22) *and* (2.23)*, is commutative.*

Proof. For $\sigma \in \mathrm{Gal}(L/L_0)$, we have $\mathrm{res}_M(\sigma) = \sigma\mid_M \in \mathrm{Gal}(M/M_0)$, because $L_0 \cap M = L \cap K^{nr} \cap M = M \cap K^{nr} = M_0$. Now, for any $\sigma \in \mathrm{Gal}(L/L_0)$, in accordance with the definition, we can write $\phi^{(\varphi^d)}_{L/L_0}(\sigma) = U_\sigma.U_{\mathbb{X}(L/L_0)}$, where

$U_\sigma \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}$ satisfies the equation $U_\sigma^{1-\varphi^d} = \Pi^{\sigma-1}_{\varphi^d;L/L_0}$. Thus, to prove the commutativity of the square, it suffices to check that

$$\widetilde{\mathcal{N}}_{L/M}(U_\sigma^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}}) \equiv U_{\sigma|_M} \pmod{U_{\mathbb{X}(M/M_0)}},$$

where $U_{\sigma|_M} \in U^{\diamond}_{\widetilde{\mathbb{X}}(M/M_0)}$ satisfies $U_{\sigma|_M}^{1-\varphi^{d'}} = \Pi^{\sigma|_M-1}_{\varphi^{d'};M/M_0}$. By Remark 2.8, without loss of generality we can fix a *basic sequence* (cf. [8])

$$L_0 = E_0 \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L,$$

where

(i) $L = \bigcup_{0\leq i\in\mathbb{Z}} E_i$;
(ii) E_i/L_0 is a Galois extension for every $0 \leq i \in \mathbb{Z}$;
(iii) E_{i+1}/E_i is cyclic of prime degree $[E_{i+1} : E_i] = p = \text{char}(\kappa_{L_0})$ for each $1 \leq i \in \mathbb{Z}$;
(iv) E_1/E_0 is cyclic of degree relatively prime to p.

Thus, each extension E_i/L_0 is finite and Galois for $0 \leq i \in \mathbb{Z}$. Now, we note that

$$\widetilde{\mathcal{N}}_{L/M}(U_\sigma^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}})^{1-\varphi^{d'}} = \widetilde{\mathcal{N}}_{L/M}(U_\sigma)^{1-\varphi^d} = \widetilde{\mathcal{N}}_{L/M}(U_\sigma^{1-\varphi^d}).$$

Since $U_\sigma^{1-\varphi^d} = \Pi^{\sigma-1}_{\varphi^d;L/L_0}$, setting $U_\sigma = (u_{\widetilde{E}_i})_{0\leq i\in\mathbb{Z}}$, for $0 \leq i \in \mathbb{Z}$ we obtain

$$\begin{aligned}
&\widetilde{\mathcal{N}}_{L/M}(U_\sigma^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}})_i^{1-\varphi^{d'}} \\
&= \widetilde{\mathcal{N}}_{L/M}\left(\Pi^{\sigma-1}_{\varphi^d;L/L_0}\right)_i = \widetilde{N}_{E_i/E_i\cap M}(\pi_{E_i}^{\sigma-1})^{1+\varphi^{d'}+\cdots+\varphi^{d'(f(L/M)-1)}} \\
&= N_{E_i/E_i\cap M}(\pi_{E_i}^{\sigma-1}) = \pi_{E_i\cap M}^{\sigma|_M-1}.
\end{aligned}$$

It follows that $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U_\sigma)^{1-\varphi^{d'}} = \Pi^{\sigma|_M-1}_{\varphi^{d'};M/M_0}$, which yields the congruence $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U_\sigma) \equiv U_{\sigma|_M} \pmod{U_{\mathbb{X}(M/M_0)}}$. This completes the proof. □

We arrive at the following theorem.

Theorem 2.11. *For an infinite Galois subextension M/K of L/K such that the residue-class degree $[\kappa_M : \kappa_K]$ is equal to d' and $K \subset M \subset K_{\varphi^{d'}}$ for some*

$d' \mid d$, the following square is commutative:

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\boldsymbol{\phi}_{L/K}^{(\varphi)}} & K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \\ \Big\downarrow \mathrm{res}_M & & \Big\downarrow \left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right) \\ \mathrm{Gal}(M/K) & \xrightarrow{\boldsymbol{\phi}_{M/K}^{(\varphi)}} & K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}, \end{array}$$

where the right vertical arrow

$$K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \xrightarrow{\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)} K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)}$$

is defined by the rule

$$\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right) : (\overline{a}, \overline{U}) \mapsto \left(e^{\mathrm{CFT}}_{L_0/M_0}(\overline{a}), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})\right)$$

for every $(\overline{a}, \overline{U}) \in K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}$. Here,

$$e^{\mathrm{CFT}}_{L_0/M_0} : K^\times/N_{L_0/K}L_0^\times \to K^\times/N_{M_0/K}M_0^\times$$

is the natural inclusion defined via the existence theorem of the local class field theory.

Proof. By the isomorphism defined by (2.1) and (2.2), for any $\sigma \in \mathrm{Gal}(L/K)$ there exists a unique $0 \leq m \in \mathbb{Z}$ such that $\sigma \mid_{L_0} = \varphi^m$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. By the definition, we have

$$\boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma) = \left(\pi_K^m N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right).$$

Thus,

$$\begin{aligned} &\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)\left(\pi_K^m N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma)\right) \\ &\quad = \left(e^{\mathrm{CFT}}_{L_0/M_0}(\pi_K^m N_{L_0/K}L_0^\times), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma))\right) \\ &\quad = \left(\pi_K^m N_{M_0/K}M_0^\times, \phi_{M/M_0}^{(\varphi^{d'})}(\varphi^{-m}\sigma \mid_M)\right) \end{aligned}$$

by Lemma 2.10. The existence theorem of the local class field theory yields

$$e^{\mathrm{CFT}}_{L_0/M_0}(\pi_K^m N_{L_0/K}L_0^\times) = \pi_K^m N_{M_0/K}M_0^\times = \pi_K^{m'} N_{M_0/K}M_0^\times,$$

where $0 \leq m' \in \mathbb{Z}$ is a unique integer satisfying $(\sigma\mid_M)\mid_{M_0} = \sigma\mid_{M_0} = \varphi^{m'}$ and $\varphi^{-m'}(\sigma\mid_M) \in \mathrm{Gal}(M/M_0)$. Hence,

$$\begin{aligned}\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)(\boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma)) &= \left(\pi_K^{m'} N_{M_0/K} M_0^\times, \phi^{(\varphi^{d'})}_{M/M_0}(\varphi^{-m}\sigma\mid_M)\right) \\ &= \left(\pi_K^{m'} N_{M_0/K} M_0^\times, \phi^{(\varphi^{d'})}_{M/M_0}(\varphi^{-m'}(\sigma\mid_M))\right) = \boldsymbol{\phi}^{(\varphi)}_{M/K}(\mathrm{res}_M(\sigma))\end{aligned}$$

by Remark 2.2, part (i), which completes the proof. □

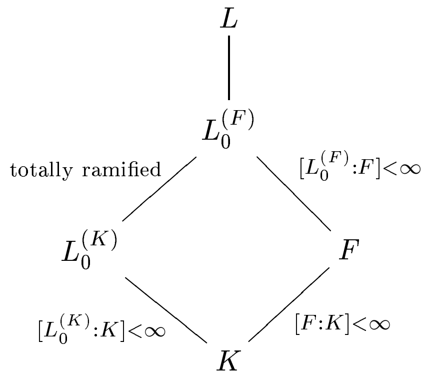
Now, let F/K be a finite subextension of L/K. Thus, L/F is an infinite APF-Galois extension (see Lemma 3.3 in [8]). We fix a Lubin–Tate splitting φ_F over F and assume that the residue-class degree $[\kappa_L : \kappa_F]$ is equal to d' for some $d' \mid d$, and that there exists a chain of field extensions

$$F \subset L \subset F_{(\varphi_F)^{d'}}.$$

Thus, we have the generalized arrow

$$\boldsymbol{\phi}^{(\varphi_F)}_{L/F} : \mathrm{Gal}(L/F) \to F^\times / N_{L_0^{(F)}/F} L_0^{(F)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)} / U_{\mathbb{X}(L/F)}$$

corresponding to the extension L/F. Here, as usual, $L_0^{(F)}$ is defined by $L_0^{(F)} = L \cap F^{nr} = F^{nr}_{d'}$. Observe the following diagram of field extensions:



Thus, $L/L_0^{(F)}$ and $L/L_0^{(K)}$ are infinite totally ramified APF-Galois extensions, by Lemma 3.3 in [8], and we have $L_0^{(F)} \subset L \subset \left(L_0^{(F)}\right)_{\varphi_F^{d'}}$ and $L_0^{(K)} \subset L \subset \left(L_0^{(K)}\right)_{\varphi_K^{d}}$.

Remark 2.12. Note that $L_0^{(F)}$ is compatible with $(L_0^{(K)}, \varphi_{L_0^{(K)}})$ in the sense of [10, pp. 89], where $\varphi_{L_0^{(K)}} = \varphi_K^d$. Thus, $\varphi_{L_0^{(F)}} = \varphi_F^{d'} = \varphi_{L_0^{(K)}}^{f(L_0^{(F)}/L_0^{(K)})} = \varphi_K^d$, because $L_0^{(F)}/L_0^{(K)}$ is totally ramified.

For the extension $L/L_0^{(F)}$, we fix an ascending chain

$$L_0^{(F)} = F_o \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset L$$

satisfying $L = \bigcup_{0 \leq i \in \mathbb{Z}} F_i$ and $[F_{i+1} : F_i] < \infty$ for every $0 \leq i \in \mathbb{Z}$. Following [8] we introduce a homomorphism

$$\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/L_0^{(F)})^\times \to \widetilde{\mathbb{X}}(L/L_0^{(K)})^\times \tag{2.25}$$

by

$$\begin{aligned} \Lambda_{F/K} : \big(\alpha_{F_0} \xleftarrow{\widetilde{N}_{F_1/F_0}} \alpha_{F_1} \xleftarrow{\widetilde{N}_{F_2/F_1}} \cdots\big) \\ \mapsto (\widetilde{N}_{L_0^{(F)}/L_0^{(K)}}(\alpha_{F_0}) \xleftarrow{\widetilde{N}_{L_0^{(F)}/L_0^{(K)}}} \alpha_{F_0} \xleftarrow{\widetilde{N}_{F_1/F}} \alpha_{F_1} \xleftarrow{\widetilde{N}_{F_2/F_1}} \cdots) \end{aligned} \tag{2.26}$$

for each $(\alpha_{F_i})_{0 \leq i \in \mathbb{Z}} \in \widetilde{\mathbb{X}}(L/L_0^{(F)})^\times$. This homomorphism induces a group homomorphism

$$\lambda_{F/K} : U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/U_{\mathbb{X}(L/L_0^{(F)})} \to U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/U_{\mathbb{X}(L/L_0^{(K)})} \tag{2.27}$$

defined by

$$\lambda_{F/K} : \overline{U} \mapsto \Lambda_{F/K}(U).U_{\mathbb{X}(L/L_0^{(K)})} \tag{2.28}$$

for every $U \in U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}$, where the symbol $\overline{U}$ denotes the coset $U.U_{\mathbb{X}(L/L_0^{(F)})}$ in $U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/U_{\mathbb{X}(L/L_0^{(F)})}$ (see [8] for the details).

Lemma 2.13. *Let F/K be a finite subextension of L/K. Fix a Lubin–Tate splitting φ_F over F. Assume that the residue-class degree $[\kappa_L : \kappa_F]$ is equal to d' and $F \subset L \subset F_{(\varphi_F)^{d'}}$ for some $d' \mid d$. Then the square*

$$\begin{array}{ccc} \mathrm{Gal}(L/L_0^{(F)}) & \xrightarrow{\phi^{(\varphi_K^d)}_{L/L_0^{(F)}}} & U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/U_{\mathbb{X}(L/L_0^{(F)})} \\ \text{inc.}\downarrow & & \downarrow \lambda_{F/K} \\ \mathrm{Gal}(L/L_0^{(K)}) & \xrightarrow{\phi^{(\varphi_K^d)}_{L/L_0^{(K)}}} & U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/U_{\mathbb{X}(L/L_0^{(K)})}, \end{array} \tag{2.29}$$

where the right vertical arrow

$$\lambda_{F/K} : U^{\diamond}_{\tilde{\mathbb{X}}(L/L_0^{(F)})}/U_{\mathbb{X}(L/L_0^{(F)})} \to U^{\diamond}_{\tilde{\mathbb{X}}(L/L_0^{(K)})}/U_{\mathbb{X}(L/L_0^{(K)})}$$

is defined by (2.27) *and* (2.28), *is commutative.*

Proof. Look at the proof of Theorem 5.12 in [8]. □

We arrive at the following theorem.

Theorem 2.14. *Let F/K be a finite subextension of L/K. Fix a Lubin–Tate splitting φ_F over F. Assume that the residue-class degree $[\kappa_L : \kappa_F]$ is equal to d' and $F \subset L \subset F_{(\varphi_F)^{d'}}$ for some $d' \mid d$. Then the following square is commutative*:

$$\begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\boldsymbol{\phi}^{(\varphi_F)}_{L/F}} & F^\times/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \\ \Big\downarrow{\scriptstyle \mathrm{inc.}} & & \Big\downarrow{\scriptstyle (N_{F/K},\lambda_{F/K})} \\ \mathrm{Gal}(L/K) & \xrightarrow{\boldsymbol{\phi}^{(\varphi_K)}_{L/K}} & K^\times/N_{L_0^{(K)}/K}L_0^{(K)\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}, \end{array} \tag{2.30}$$

where the right vertical arrow

$$\begin{aligned}(N_{F/K},\lambda_{F/K}) : F^\times/N_{L_0^{(F)}/F}L_0^{(F)\times} &\times U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)} \\ \to K^\times/N_{L_0^{(K)}/K}L_0^{(K)\times} &\times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)}\end{aligned}$$

is defined by

$$(N_{F/K},\lambda_{F/K}) : (\overline{a},\overline{U}) \mapsto \left(\overline{N_{F/K}(a)}, \lambda_{F/K}(\overline{U})\right)$$

for every $(\overline{a},\overline{U}) \in F^\times/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\tilde{\mathbb{X}}(L/F)}/U_{\mathbb{X}(L/F)}$.

Proof. Let $\sigma \in \mathrm{Gal}(L/F)$. There exists $0 \le m \in \mathbb{Z}$ such that $\sigma\mid_{L_0^{(F)}} = \varphi_F^m$ and $\varphi_F^{-m}\sigma \in \mathrm{Gal}(L/L_0^{(F)})$. Now,

$$\boldsymbol{\phi}^{(\varphi_F)}_{L/F}(\sigma) = \left(\pi_F^m.N_{L_0^{(F)}/F}L_0^{(F)\times}, \phi^{(\varphi_K^d)}_{L/L_0^{(F)}}(\varphi_F^{-m}\sigma)\right)$$

and

$$(N_{F/K},\lambda_{F/K})(\boldsymbol{\phi}^{(\varphi_F)}_{L/F}(\sigma)) = \left(\pi_K^m.N_{L_0^{(K)}/K}L_0^{(K)\times}, \phi^{(\varphi_K^d)}_{L/L_0^{(K)}}(\varphi_F^{-m}\sigma)\right),$$

by the norm-compatibility of primes in the fixed Lubin–Tate labeling and by Lemma 2.13. There exists $0 \le m' \in \mathbb{Z}$ such that $\sigma\mid_{L_0^{(K)}} = \varphi_K^{m'}$ and $\varphi_K^{-m'}\sigma \in$

$\mathrm{Gal}(L/L_0^{(K)})$. By part (ii) of Remark 2.2, it follows that $\varphi_F^m \mid_{L_0^{(K)}} = \varphi_K^{m'}$ and $\varphi_F^{-m}\sigma = \varphi_K^{-m'}\sigma$. Now, $N_{F/K} : \pi_F^m N_{L_0^{(F)}/F} L_0^{(F)\times} \mapsto \pi_K^{m'} N_{L_0^{(K)}/K} L_0^{(K)\times} = \pi_K^m . N_{L_0^{(K)}/K} L_0^{(K)\times}$ by the Abelian local class field theory. Thus,

$$(N_{F/K}, \lambda_{F/K})(\boldsymbol{\phi}_{L/F}^{(\varphi_F)}(\sigma)) = \left(\pi_K^m . N_{L_0^{(K)}/K} L_0^{(K)\times}, \phi_{L/L_0^{(K)}}^{(\varphi_K^d)}(\varphi_F^{-m}\sigma)\right)$$

$$= \left(\pi_K^{m'} . N_{L_0^{(K)}/K} L_0^{(K)\times}, \phi_{L/L_0^{(K)}}^{(\varphi_K^d)}(\varphi_K^{-m'}\sigma)\right) = \boldsymbol{\phi}_{L/K}^{(\varphi_K)}(\sigma),$$

which completes the proof. □

Let L/K be any APF-Galois subextension of K_{φ^d}/K, where the residue-class degree is d. If L/K is assumed to be a finite extension, then the $\widetilde{K}$-coordinate of the generalized arrow $\boldsymbol{\phi}_{L/K}^{(\varphi)} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}/U_{\mathbb{X}(L/K)}$ is the Iwasawa–Neukirch map $\iota_{L/K}$ of L/K (for the details on the Iwasawa–Neukirch map $\iota_{L/K}$ of the Galois extension L/K, see [8, §1]). More precisely, we have the following statement.

Proposition 2.15. *Define a homomorphism*

$$\rho : K^\times/N_{L_0/K}L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}/U_{\mathbb{X}(L/K)} \to K^\times/N_{L/K}L^\times \tag{2.31}$$

by

$$\rho : (\pi_K^m, (u_{\widetilde{E}})) \mapsto \pi_K^m N_{L_0/K}(u_{\widetilde{L}_0}) \mod N_{L/K}L^\times \tag{2.32}$$

for every $(\pi_K^m, (u_{\widetilde{E}})) \in K^\times/N_{L_0/K}L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}/U_{\mathbb{X}(L/K)}$. *Then the composite map*

$$\mathrm{Gal}(L/K) \xrightarrow{\boldsymbol{\phi}_{L/K}^{(\varphi)}} K^\times/N_{L_0/K}L_0^\times \times U_{\widetilde{\mathbb{X}}(L/K)}^{\diamond}/U_{\mathbb{X}(L/K)} \xrightarrow{\rho} K^\times/N_{L/K}L^\times, \qquad \rho\circ\boldsymbol{\phi}_{L/K}^{(\varphi)} = \iota_{L/K} \tag{2.33}$$

is the Iwasawa–Neukirch map $\iota_{L/K} : Gal(L/K) \to K^\times/N_{L/K}L^\times$ *of* L/K.

Proof. We follow §1 in [8] to briefly recall the construction of the Iwasawa–Neukirch map

$$\iota_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L/K}L^\times$$

for the Galois extension L/K. For every $\sigma \in \mathrm{Gal}(L/K)$, we choose $\sigma^* \in \mathrm{Gal}(L^{nr}/K)$ in such a way that

(i) $\sigma^* \mid_L = \sigma$; and

(ii) $\sigma^* \mid_{K^{nr}} = \varphi^n$ for some $0 < n \in \mathbb{Z}$.

Let Σ_{σ^*} be the fixed field $(L^{nr})^{\sigma^*}$ of $\sigma^* \in \mathrm{Gal}(L^{nr}/K)$ in L^{nr}. It is well known that $[\Sigma_{\sigma^*} : K] < \infty$. Now, the map $\iota_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L/K}L^\times$ is defined by $\iota_{L/K}(\sigma) = N_{\Sigma_{\sigma^*}/K}(\pi_{\Sigma_{\sigma^*}}) \mod N_{L/K}L^\times$, for $\sigma \in \mathrm{Gal}(L/K)$, where $\pi_{\Sigma_{\sigma^*}}$ denotes any prime element of Σ_{σ^*}. Thus, for a finite APF-Galois extension L/K satisfying $[\kappa_L : \kappa_K] = d$ and $K \subset L \subset K_{\varphi^d}$, it suffices to prove that, for $\sigma \in \mathrm{Gal}(L/K)$,

$$\rho \circ \boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma) = \iota_{L/K}(\sigma) = N_{\Sigma_{\sigma^*}/K}(\pi_{\Sigma_{\sigma^*}}) \mod N_{L/K}L^\times,$$

where $\pi_{\Sigma_{\sigma^*}}$ denotes any prime element of Σ_{σ^*}. For $\sigma \in \mathrm{Gal}(L/K)$, there exists $0 \le m \in \mathbb{Z}$ such that $\sigma \mid_{L_0} = \varphi^m$ and $\tau = \varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. Thus, $\sigma = \varphi^m\tau$.

Case 1: $m > 0$. In this case, it suffices to prove that

$$\pi_K^m N_{L_0/K}\left(\mathrm{Pr}_{\widetilde{L}_0}(\phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma))\right) = N_{\Sigma_{\sigma^*}/K}(\pi_{\Sigma_{\sigma^*}}) \mod N_{L/K}L^\times,$$

where $\pi_{\Sigma_{\sigma^*}}$ denotes any prime element of Σ_{σ^*}. To prove this relation, we choose $\sigma^* \in \mathrm{Gal}(L^{nr}/K)$ such that

(i) $\sigma^* \mid_L = \sigma$;

(ii) $\sigma^* \mid_{K^{nr}} = \varphi^m$.

Indeed, let $\sigma^* = \varphi^m \mid_{L^{nr}} \tau^*$, where $\tau^* \in \mathrm{Gal}(L^{nr}/L_0)$ is defined uniquely by the conditions $\tau^* \mid_L = \tau$ and $\tau^* \mid_{K^{nr}} = \mathrm{id}_{K^{nr}}$, because $L^{nr} = LK^{nr}$. Note that, for $\Sigma = \Sigma_{\sigma^*}$, $\Sigma_0 = \Sigma \cap K^{nr}$ is a finite extension of degree $[\Sigma_0 : K] = m$ over K, because Σ_0 coincides with $(K^{nr})^{\varphi^m}$, the fixed field of $\varphi^m \in \mathrm{Gal}(K^{nr}/K)$ in K^{nr}. Since L is fixed by φ^d, $T = L \cap K_\varphi$ is an unramified extension of degree d. Thus, the prime element π_T is a prime element of L and of L^{nr}. Now, choose a prime element π_Σ of Σ. It is well known that $\Sigma^{nr} = L^{nr}$ (see §2 in Chapter 4 of [4]). Thus, π_Σ is a prime element of L^{nr}. So, there exists a unit $v \in L^{nr} \subset \widetilde{L}$ such that $\pi_\Sigma = \pi_T v$. Note that $\pi_\Sigma^{\sigma^*-1} = 1$ because Σ is fixed by σ^*. Thus, $(\pi_T v)^{\sigma^*-1} = 1$ and we get the relations

$$\pi_T^{\sigma-1} = v^{1-\sigma^*} = v^{1-\varphi^m\tau^*} = v^{1-\tau^*}v^{(1-\varphi^m)\tau^*}.$$

Recall that (by Proposition 1.8 in Chapter IV of [4] or by Subsection 1.1 in [10]), $U_{\widetilde{L}}$ is multiplicatively $(1-\varphi^m)$-divisible. So, there exists $w \in U_{\widetilde{L}}$ such that $w^{1-\varphi^m} = v$. Hence,

$$\pi_T^{\sigma-1} = (w^{1-\tau^*}v^{\tau^*})^{1-\varphi^m},$$

because $\varphi^m\tau^* = \tau^*\varphi^m$. Now, we choose $z \in U_{\widetilde{L}}$ in the following way: $z = (w^{1-\tau^*}v^{\tau^*})^{1+\varphi+\cdots+\varphi^{m-1}}$. Observe that this $z \in U_{\widetilde{L}}$ satisfies $z^{1-\varphi} = \pi_T^{\sigma-1}$. Clearly,

$$\widetilde{N}_{L/K}(z) = \widetilde{N}_{L/K}(v)^{1+\varphi+\cdots+\varphi^{m-1}}.$$

After these preliminary observations, we see that

$$\begin{aligned} N_{\Sigma/K}(\pi_\Sigma) &= N_{\Sigma_0/K} \circ N_{\Sigma/\Sigma_0}(\pi_\Sigma) \\ &= \widetilde{N}_{L/K}(\pi_\Sigma)^{1+\varphi+\cdots+\varphi^{m-1}} \\ &= \widetilde{N}_{L/K}(\pi_T v)^{1+\sigma+\cdots+\sigma^{m-1}} \\ &= \pi_K^m \widetilde{N}_{L/K}(v)^{1+\varphi+\cdots+\varphi^{m-1}} \\ &= \pi_K^m \widetilde{N}_{L/K}(z), \end{aligned}$$

because π_T belongs to the fixed Lubin–Tate labeling. Thus, the image of σ under the Iwasawa–Neukirch map $\iota_{L/K}$ is

$$\iota_{L/K}(\sigma) = \pi_K^m \widetilde{N}_{L/K}(z) \mod N_{L/K}L^\times.$$

Now, let $y \in U_L$ be such that

$$y^{1-\varphi^d} = \pi_T^{\varphi^{-m}\sigma-1} = \pi_T^{\sigma-1}.$$

Note that $T = L \cap K_\varphi$. Then, setting $z = y^{1+\varphi+\cdots+\varphi^{d-1}} \in U_L$, we have

$$z^{1-\varphi} = y^{1-\varphi^d} = \pi_T^{\sigma-1}.$$

Thus,

$$N_{L/K}(y) = \widetilde{N}_{L/K}(y)^{1+\varphi+\cdots+\varphi^{d-1}} = \widetilde{N}_{L/K}(z),$$

which shows that

$$\begin{aligned} \iota_{L/K}(\sigma) &= \pi_K^m \widetilde{N}_{L/K}(z) \mod N_{L/K}L^\times \\ &= \pi_K^m N_{L/K}(y) \mod N_{L/K}L^\times \\ &= \pi_K^m N_{L_0/K}\left(\widetilde{N}_{L/K}(y)\right) \mod N_{L/K}L^\times \\ &= \pi_K^m N_{L_0/K}\left(\mathrm{Pr}_{\widetilde{L}_0}(\phi_{L/L_0}^{(\varphi^d)}(\varphi^{-m}\sigma))\right) \\ &= \rho \circ \boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma), \end{aligned}$$

completing the proof.

Case 2: $m = 0$. In this case, $\sigma \in \mathrm{Gal}(L/L_0)$. Consider $\varphi^d\sigma \in \mathrm{Gal}(L^{nr}/K)$. Then, by Case 1, we have

$$\iota_{L/K}(\varphi^d\sigma) = \rho \circ \boldsymbol{\phi}_{L/K}^{(\varphi)}(\varphi^d\sigma),$$

where $\iota_{L/K}(\varphi^d\sigma) = \iota_{L/K}(\sigma)$. Theorem 2.4 implies the formulas

$$\boldsymbol{\phi}_{L/K}^{(\varphi)}(\varphi^d\sigma) = \boldsymbol{\phi}_{L/K}^{(\varphi)}(\varphi^d)\boldsymbol{\phi}_{L/K}^{(\varphi)}(\sigma)^{\varphi^d} = \left(\pi_K^d.N_{L_0/K}L_0^\times, \phi_{L/L_0}^{(\varphi^d)}(\sigma)\right),$$

where the last identity follows from the fact that $K \subset L \subset K_{\varphi^d}$. Thus,

$$\rho \circ \boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma) = \rho \circ \boldsymbol{\phi}^{(\varphi)}_{L/K}(\varphi^d \sigma) = \pi_K^d N_{L_0/K}\left(\mathrm{Pr}_{\widetilde{L}_0}(\phi^{(\varphi^d)}_{L/L_0}(\sigma))\right) \mod N_{L/K}L^\times,$$

which proves the relation

$$\iota_{L/K}(\sigma) = \rho \circ \boldsymbol{\phi}^{(\varphi)}_{L/K}(\sigma).$$

The proposition is proved. □

Now, we generalize the definition of the extended Hazewinkel map $H^{(\varphi)}_{L/K} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \to \mathrm{Gal}(L/K)$ of Fesenko (cf. [1, 2, 3] and [8]), initially defined for totally ramified APF-Galois subextensions L/K of K_φ/K, to infinite APF-Galois subextensions L/K of K_{φ^d}/K, where $[\kappa_L : \kappa_K] = d$.

For this, first we assume that the local field K satisfies the condition

$$\boldsymbol{\mu}_p(K^{\mathrm{sep}}) = \{\alpha \in K^{\mathrm{sep}} : \alpha^p = 1\} \subset K, \tag{2.34}$$

where $p = \mathrm{char}(\kappa_K)$. For the details on the assumption (2.34) on K, we refer the reader to [1, 2, 3].

Let L/K be an infinite APF-Galois extension with residue-class degree $[\kappa_L : \kappa_K] = d$ and with $K \subset L \subset K_{\varphi^d}$. As usual, let $L_0 = L \cap K^{nr}$. We introduce the generalized arrow

$$\boldsymbol{H}^{(\varphi)}_{L/K} : K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to \mathrm{Gal}(L/K) \tag{2.35}$$

for the extension L/K by

$$\boldsymbol{H}^{(\varphi)}_{L/K}\left((\pi_K^m N_{L_0/K}L_0^\times, U.Y_{L/L_0})\right) = \varphi^m \mid_L H^{(\varphi^d)}_{L/L_0}(U.Y_{L/L_0}) \tag{2.36}$$

for every $m \in \mathbb{Z}$ and every $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where $H^{(\varphi^d)}_{L/L_0} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to \mathrm{Gal}(L/L_0)$ is the extended Hazewinkel map of Fesenko for the extension L/L_0. For the definition and basic properties of the group Y_{L/L_0}, we refer the reader to [3] and [8].

The following lemma is clear.

Lemma 2.16. *Suppose that the local field K satisfies condition* (2.34)*. Let L/K be an infinite APF-Galois subextension of K_{φ^d}/K, where $d = [\kappa_L : \kappa_K]$. Then the generalized arrow*

$$\boldsymbol{H}^{(\varphi)}_{L/K} : K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to \mathrm{Gal}(L/K) \tag{2.37}$$

for the extension L/K is a bijection.

Proof. The proof follows from the isomorphism

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(L_0/K) \times \mathrm{Gal}(L/L_0)$$

combined with the bijectivity of

$$H^{(\varphi^d)}_{L/L_0} : U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to \mathrm{Gal}(L/L_0)$$

(cf. Lemma 5.22 of [8]) and the Abelian local class field theory for the extension L_0/K. □

Now, consider the composition of arrows

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\boldsymbol{\phi}^{(\varphi)}_{L/K}} & K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \\ & \searrow^{\boldsymbol{\Phi}^{(\varphi)}_{L/K}} & \Big\downarrow (\mathrm{id}_{K^\times/N_{L_0/K}L_0^\times}, c_{L/L_0}) \\ & & K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \end{array} \tag{2.38}$$

where $c_{L/L_0} : U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is the canonical map defined via the inclusion $U_{\mathbb{X}(L/K)} \subseteq Y_{L/L_0}$. Recall that (see [8, (5.35)]) the composition $c_{L/L_0} \circ \phi^{(\varphi^d)}_{L/L_0} = \Phi^{(\varphi^d)}_{L/L_0} : \mathrm{Gal}(L/L_0) \to U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is the Fesenko reciprocity map for the extension L/L_0. Now, let $\sigma \in \mathrm{Gal}(L/K)$. Let $0 \le m \in \mathbb{Z}$ be such that $\sigma \mid_{L_0} = \varphi^m \mid_{L_0}$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. Then, in accordance with the definition,

$$\begin{aligned} \boldsymbol{H}^{(\varphi)}_{L/K} \circ \boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma) &= \boldsymbol{H}^{(\varphi)}_{L/K}\left(\pi_K^m.N_{L_0/K}L_0^\times, c_{L/L_0} \circ \phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right) \\ &= \boldsymbol{H}^{(\varphi)}_{L/K}\left(\pi_K^m.N_{L_0/K}L_0^\times, \Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right) \\ &= \varphi^m \mid_L H^{(\varphi^d)}_{L/L_0}\left(\Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right) \\ &= \varphi^m \mid_L (\varphi^{-m}\sigma) \\ &= \sigma \end{aligned}$$

by [8, Lemma 5.23]. For $0 \le m \in \mathbb{Z}$ and $U \in U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}$, let

$$(\pi_K^m.N_{L_0/K}L_0^\times, U.Y_{L/L_0}) \in K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}.$$

Now, again by definition,

$$\begin{aligned}
&\mathbf{\Phi}^{(\varphi)}_{L/K} \circ \boldsymbol{H}^{(\varphi)}_{L/K} \left((\pi_K^m.N_{L_0/K}L_0^\times, U.Y_{L/L_0})\right) = \mathbf{\Phi}^{(\varphi)}_{L/K} \left(\varphi^m \mid_L H^{(\varphi^d)}_{L/L_0}(U.Y_{L/L_0})\right) \\
&= (\mathrm{id}_{K^\times/N_{L_0/K}L_0^\times}, c_{L/L_0}) \circ \boldsymbol{\phi}^{(\varphi)}_{L/K} \left(\varphi^m \mid_L H^{(\varphi^d)}_{L/L_0}(U.Y_{L/L_0})\right) \\
&= (\mathrm{id}_{K^\times/N_{L_0/K}L_0^\times}, c_{L/L_0}) \left(\pi_K^m.N_{L_0/K}L_0^\times, \phi^{(\varphi^d)}_{L/L_0}(H^{(\varphi^d)}_{L/L_0}(U.Y_{L/L_0})\right) \\
&= \left(\pi_K^m.N_{L_0/K}L_0^\times, \Phi^{(\varphi^d)}_{L/L_0}(H^{(\varphi^d)}_{L/L_0}(U.Y_{L/L_0})\right) \\
&= \left(\pi_K^m.N_{L_0/K}L_0^\times, U.Y_{L/L_0}\right),
\end{aligned}$$

by [8, Lemma 5.23]. These computations yield

$$\boldsymbol{H}^{(\varphi)}_{L/K} \circ \mathbf{\Phi}^{(\varphi)}_{L/K} = \mathrm{id}_{\mathrm{Gal}(L/K)} \tag{2.39}$$

and

$$\mathbf{\Phi}^{(\varphi)}_{L/K} \circ \boldsymbol{H}^{(\varphi)}_{L/K} = \mathrm{id}_{K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}}. \tag{2.40}$$

Note that there is a natural continuous action of $\mathrm{Gal}(L/K)$ on the topological group $K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$, defined by the Abelian local class field theory on the first component and by formula (5.7) and Lemma 5.20 in [8] on the second component:

$$(\overline{a}, \overline{U})^\sigma = \left(\overline{a}^{\varphi^m}, \overline{U}^{\varphi^{-m}\sigma}\right) = \left(\overline{a}, \overline{U}^{\varphi^{-m}\sigma}\right), \tag{2.41}$$

for every $\sigma \in \mathrm{Gal}(L/K)$, where $\sigma \mid_{L_0} = \varphi^m$ for some $0 \leq m \in \mathbb{Z}$ and for every $a \in K^\times$ with $\overline{a} = a.N_{L_0/K}L_0^\times$ and $U \in U^\diamond_{\tilde{\mathbb{X}}(L/K)}$ with $\overline{U} = U.Y_{L/L_0}$. *We shall always view* $K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ *as a topological* $\mathrm{Gal}(L/K)$*-module in this paper.*

So, we arrive at the following theorem, which follows from Theorem 2.4, Lemma 2.16, and relations (2.39) and (2.40) combined with the fact that $U_{\mathbb{X}(L/K)}$ is a topological $\mathrm{Gal}(L/L_0)$-submodule of Y_{L/L_0}.

Theorem 2.17. *Suppose that the local field K satisfies condition* (2.34)*. Let L/K be an infinite APF-Galois subextension of K_{φ^d}/K, where $d = [\kappa_L : \kappa_K]$. The mapping*

$$\mathbf{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$$

defined for the extension L/K is a bijection with the inverse

$$\boldsymbol{H}^{(\varphi)}_{L/K} : K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to \mathrm{Gal}(L/K).$$

For every $\sigma, \tau \in \mathrm{Gal}(L/K)$, the cocycle condition

$$\mathbf{\Phi}^{(\varphi)}_{L/K}(\sigma\tau) = \mathbf{\Phi}^{(\varphi)}_{L/K}(\sigma)\mathbf{\Phi}^{(\varphi)}_{L/K}(\tau)^{\sigma} \tag{2.42}$$

is satisfied.

By Corollary 2.5, Theorem 2.17 has the following consequence.

Corollary 2.18. *Let a law of composition $*$ be defined on $K^{\times}/N_{L_0/K}L_0^{\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ by the rule*

$$(\overline{a}, \overline{U}) * (\overline{b}, \overline{V}) = (\overline{a}, \overline{U}).(\overline{b}, \overline{V})^{(\mathbf{\Phi}^{(\varphi)}_{L/K})^{-1}((\overline{a},\overline{U}))} \tag{2.43}$$

for every $\overline{a} = a.N_{L_0/K}L_0^{\times}, \overline{b} = b.N_{L_0/K}L_0^{\times} \in K^{\times}/N_{L_0/K}L_0^{\times}$ with $a, b \in K^{\times}$ and every $\overline{U} = U.Y_{L/L_0}, \overline{V} = V.Y_{L/L_0} \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ with $U, V \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$. Then $K^{\times}/N_{L_0/K}L_0^{\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is a topological group under $*$, and the map $\mathbf{\Phi}^{(\varphi)}_{L/K}$ induces an isomorphism of topological groups

$$\mathbf{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} K^{\times}/N_{L_0/K}L_0^{\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}, \tag{2.44}$$

where the topological group structure on $K^{\times}/N_{L_0/K}L_0^{\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$ is defined with respect to the binary operation $*$ introduced by (2.43).

Definition 2.19. Let K be a local field satisfying condition (2.34). Let L/K be an infinite APF-Galois subextension of K_{φ^d}/K, where $d = [\kappa_L : \kappa_K]$. The mapping

$$\mathbf{\Phi}^{(\varphi)}_{L/K} : \mathrm{Gal}(L/K) \to K^{\times}/N_{L_0/K}L_0^{\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$$

defined in Theorem 2.17 is called the *generalized Fesenko reciprocity map for the extension L/K.*

We recall that, for each $0 \leq i \in \mathbb{R}$, the higher unit groups $\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^{i}$ of the field $\widetilde{\mathbb{X}}(L/K)$ were introduced in (2.16). As in [8, (5.42)], for each $0 \leq n \in \mathbb{Z}$ we put

$$Q^{n}_{L/L_0} = c_{L/L_0}\left(\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^{n} U_{\mathbb{X}(L/K)}/U_{\mathbb{X}(L/K)} \cap \mathrm{im}(\phi^{(\varphi^d)}_{L/L_0})\right); \tag{2.45}$$

this is a subgroup of $\left(U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}\right)^{n} Y_{L/L_0}/Y_{L/L_0}$. Now, the ramification statement of Theorem 2.7 can be reformulated for the generalized reciprocity map $\mathbf{\Phi}^{(\varphi)}_{L/K}$ corresponding to the extension L/K as follows.

Theorem 2.20 (Ramification theorem)**.** *Let K be a local field satisfying condition* (2.34)*. For $0 \leq u \in \mathbb{R}$, let $\mathrm{Gal}(L/K)_u$ denote the uth higher ramification subgroup in the lower numbering of the Galois group $\mathrm{Gal}(L/K)$ corresponding to the infinite APF-Galois subextension L/K of K_{φ^d}/K with residue-class degree $[\kappa_L : \kappa_K]$ equal to d. Then, for $0 \leq n \in \mathbb{Z}$, we have*

$$\begin{aligned}\mathbf{\Phi}^{(\varphi)}_{L/K}&\left(\mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n)} - \mathrm{Gal}(L/K)_{\psi_{L/K}\circ\varphi_{L/L_0}(n+1)}\right)\\ &\subseteq \left\langle 1_{K^\times/N_{L_0/K}L_0^\times}\right\rangle \times \left(\left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/L_0}/Y_{L/L_0} - Q^{n+1}_{L/L_0}\right).\end{aligned}$$

Proof. In accordance with the general observation made in the first paragarph of the proof of Theorem 2.7, for $0 \leq u \in \mathbb{R}$ and for $\tau \in \mathrm{Gal}(L/K)_u = \mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)}$ we have

$$\boldsymbol{\phi}^{(\varphi)}_{L/K}(\tau) = \left(1_{K^\times/N_{L_0/K}L_0^\times}, \phi^{(\varphi')}_{L/L_0}(\tau)\right),$$

where $\varphi' = \varphi^d$. Thus, by the definition,

$$\begin{aligned}\mathbf{\Phi}^{(\varphi)}_{L/K}(\tau) &= \left(1_{K^\times/N_{L_0/K}L_0^\times}, c_{L/L_0}\circ\phi^{(\varphi')}_{L/L_0}(\tau)\right)\\ &= \left(1_{K^\times/N_{L_0/K}L_0^\times}, \Phi^{(\varphi')}_{L/L_0}(\tau)\right).\end{aligned}$$

Now, to prove the theorem, we take $u = \psi_{L/K}\circ\varphi_{L/L_0}(n)$ and $u' = \psi_{L/K}\circ\varphi_{L/L_0}(n+1)$. Then, the ramification theorem (see [8, Theorem 5.27]) shows that, for any $\tau \in \mathrm{Gal}(L/K)_u - \mathrm{Gal}(L/K)_{u'}$ the second coordinate of $\mathbf{\Phi}^{(\varphi)}_{L/K}(\tau)$ satisfies

$$\Phi^{(\varphi')}_{L/L_0}(\tau) \in \left(U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}\right)^n Y_{L/L_0}/Y_{L/L_0} - Q^{n+1}_{L/L_0},$$

because

$$\mathrm{Gal}(L/K)_u = \mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u)} = \mathrm{Gal}(L/L_0)^{\varphi_{L/L_0}(n)} = \mathrm{Gal}(L/L_0)_n$$

and likewise

$$\mathrm{Gal}(L/K)_{u'} = \mathrm{Gal}(L/L_0)^{\varphi_{L/K}(u')} = \mathrm{Gal}(L/L_0)^{\varphi_{L/L_0}(n+1)} = \mathrm{Gal}(L/L_0)_{n+1},$$

which completes the proof. □

Let K be a local field satisfying condition (2.34). Let M/K be an infinite Galois subextension of L/K. Thus, by [8, Lemma 3.3], M is an APF-Galois extension over K. We further assume that the residue-class degree $[\kappa_M : \kappa_K]$ is equal to d' and $K \subset M \subset K_{\varphi^{d'}}$ for some $d' \mid d$. Let

$$\mathbf{\Phi}^{(\varphi)}_{M/K} : \mathrm{Gal}(M/K) \to K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(M/K)}/Y_{M/M_0}$$

be the corresponding generalized Fesenko reciprocity map defined for the extension M/K. Here, as usual, M_0 is defined by $M_0 = M \cap K^{nr} = K^{nr}_{d'}$. Now, we fix a basic sequence

$$L_o = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

for the extension L/L_0. Using the notation of [3] and [8], for each $1 \leq i \in \mathbb{Z}$ we introduce an element σ_i in $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$ such that $\langle \sigma \mid_{E_i} \rangle = \mathrm{Gal}(E_i/E_{i-1})$. Next, for each $1 \leq k, i \in \mathbb{Z}$, we introduce the map

$$h_k^{(L/L_o)} : \prod_{1 \leq i \leq k} U_{\widetilde{E}_k}^{\sigma_i - 1} \to \left(\prod_{1 \leq i \leq k+1} U_{\widetilde{E}_{k+1}}^{\sigma_i - 1} \right) \Big/ U_{\widetilde{E}_{k+1}}^{\sigma_{k+1} - 1},$$

the map

$$g_k^{(L/L_o)} : \prod_{1 \leq i \leq k} U_{\widetilde{E}_k}^{\sigma_i - 1} \to \prod_{1 \leq i \leq k+1} U_{\widetilde{E}_{k+1}}^{\sigma_i - 1}$$

and the map

$$f_i^{(L/L_o)} : U_{\widetilde{E}_i}^{\sigma_i - 1} \to U_{\widetilde{\mathbb{X}}(L/E_i)} \xrightarrow{\Lambda_{E_i/E_o}} U_{\widetilde{\mathbb{X}}(L/K)}$$

as in [3] and [8]. We fix the sequence

$$M_o = E_o \cap M \subseteq E_1 \cap M \subseteq \cdots \subseteq E_i \cap M \subseteq \cdots \subseteq M$$

for the extension M/M_o and, for each $1 \leq k \in \mathbb{Z}$, define a homomorphism

$$h_k^{(M/M_o)} : \prod_{1 \leq i \leq k} U_{\widetilde{E_k \cap M}}^{\sigma_i |_{\widetilde{M}} - 1} \to \left(\prod_{1 \leq i \leq k+1} U_{\widetilde{E_{k+1} \cap M}}^{\sigma_i |_{\widetilde{M}} - 1} \right) \Big/ U_{\widetilde{E_{k+1} \cap M}}^{\sigma_{k+1} |_{\widetilde{M}} - 1}$$

that satisfies

$$\left(\prod_{0 \leq \ell \leq f(L/M) - 1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_{k+1}/E_{k+1} \cap M} \right) \circ h_k^{(L/L_o)} = h_k^{(M/M_o)} \circ \left(\prod_{0 \leq \ell \leq f(L/M) - 1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_k/E_k \cap M} \right)$$

and take any map

$$g_k^{(M/M_o)} : \prod_{1 \leq i \leq k} U_{\widetilde{E_k \cap M}}^{\sigma_i |_{\widetilde{M}} - 1} \to \prod_{1 \leq i \leq k+1} U_{\widetilde{E_{k+1} \cap M}}^{\sigma_i |_{\widetilde{M}} - 1}$$

that satisfies

$$\left(\prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_{k+1}/E_{k+1}\cap M}\right)\circ g_k^{(L/L_o)} = g_k^{(M/M_o)}\circ\left(\prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_k/E_k\cap M}\right),$$

again following the same lines of [3] and [8].

Now, for each $1\le i\in\mathbb{Z}$, we introduce a map

$$f_i^{(M/M_o)}: U_{\widetilde{E_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1}\to U_{\widetilde{\mathbb{X}}(M/K)}$$

by

$$f_i^{(M/M_o)}(w)=\widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(v)\right),$$

where $v\in U_{\widetilde{E}_i}^{\sigma_i-1}$ is any element satisfying $\prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}(v)=w\in U_{\widetilde{E_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1}$. Observe that if $v'\in U_{\widetilde{E}_i}^{\sigma_i-1}$ is such that

$$\prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}(v')=w,$$

then $\widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(v)\right)=\widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(v')\right)$.

Indeed, there exists $u\in\ker\left(\prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}\right)$ such that $v'=vu$. Thus, we need to verify that $\widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(v)\right)=\widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(vu)\right)$. That is, for each $1\le j\in\mathbb{Z}$, we need to check the relation

$$\prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_j/E_j\cap M}\left(\mathrm{Pr}_{\widetilde{E}_j}(f_i^{(L/L_o)}(v))\right) = \prod_{0\le\ell\le f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_j/E_j\cap M}\left(\mathrm{Pr}_{\widetilde{E}_j}(f_i^{(L/L_o)}(vu))\right).$$

For $j > i$, we have

$$\begin{aligned}
&\prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_j/E_j\cap M}\left(\mathrm{Pr}_{\widetilde{E}_j}(f_i^{(L/L_o)}(v))\right)\\
&= \prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_j/E_j\cap M}\left(g_{j-1}^{(L/L_o)}\circ\cdots\circ g_i^{(L/L_o)}(v)\right)\\
&= g_{j-1}^{(M/M_o)}\circ\cdots\circ g_i^{(M/M_o)}\left(\prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_i/E_i\cap M}(v)\right)\\
&= g_{j-1}^{(M/M_o)}\circ\cdots\circ g_i^{(M/M_o)}\left(\prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_i/E_i\cap M}(vu)\right)\\
&= \prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_j/E_j\cap M}\left(g_{j-1}^{(L/L_o)}\circ\cdots\circ g_i^{(L/L_o)}(vu)\right)\\
&= \prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_j/E_j\cap M}\left(\mathrm{Pr}_{\widetilde{E}_j}(f_i^{(L/L_o)}(vu))\right).
\end{aligned}$$

Thus, the map

$$f_i^{(M/M_o)} : U_{\widetilde{E_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1} \to U_{\widetilde{\mathbb{X}}(M/K)}$$

is well defined. Moreover, for $j > i$ we have

$$\mathrm{Pr}_{\widetilde{E_j\cap M}}\circ f_i^{(M/M_o)} = \left(g_{j-1}^{(M/M_o)}\circ\cdots\circ g_i^{(M/M_o)}\right)|_{U_{\widetilde{E_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1}}.$$

Indeed, for $w \in U_{\widetilde{E_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1}$, there exists $v \in U_{\widetilde{E}_i}^{\sigma_i-1}$ such that

$$\prod_{0\leq \ell \leq f(L/M)-1} (\varphi^{d'})^{\ell} \widetilde{N}_{E_i/E_i\cap M}(v) = w,$$

and

$$f_i^{(M/M_o)}(w) = \widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(v)\right).$$

That is, the square

$$\begin{array}{ccc}
U_{\widetilde{E}_i}^{\sigma_i-1} & \xrightarrow{f_i^{(L/L_o)}} & U_{\widetilde{\mathbb{X}}(L/K)}\\
{\scriptstyle \prod_{0\leq \ell\leq f(L/M)-1}(\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}_{L/M}}\\
U_{\widetilde{E_i\cap M}}^{\sigma_i|_{\widetilde{M}}-1} & \xrightarrow{f_i^{(M/M_o)}} & U_{\widetilde{\mathbb{X}}(M/K)}
\end{array} \tag{2.46}$$

is commutative. Thus,

$$\begin{aligned}
\mathrm{Pr}_{\widetilde{E_j\cap M}} \circ f_i^{(M/M_o)}(w) &= \mathrm{Pr}_{\widetilde{E_j\cap M}} \circ \widetilde{\mathcal{N}}_{L/M}\left(f_i^{(L/L_o)}(v)\right) \\
&= \prod_{0\leq \ell\leq f(L/M)-1} (\varphi^{d'})^{\ell}\widetilde{N}_{E_j/E_j\cap M}\left(\mathrm{Pr}_{\widetilde{E}_j}\circ f_i^{(L/L_o)}(v)\right) \\
&= \prod_{0\leq \ell\leq f(L/M)-1} (\varphi^{d'})^{\ell}\widetilde{N}_{E_j/E_j\cap M}\left((g_{j-1}^{(L/L_o)}\circ\cdots\circ g_i^{(L/L_o)})(v)\right) \\
&= \left(g_{j-1}^{(M/M_o)}\circ\cdots\circ g_i^{(M/M_o)}\right)\left(\prod_{0\leq \ell\leq f(L/M)-1} (\varphi^{d'})^{\ell}\widetilde{N}_{E_i/E_i\cap M}(v)\right),
\end{aligned}$$

which is the desired relation.

Now, we modify Lemma 5.28 of [8] and show that the norm map $\widetilde{\mathcal{N}}_{L/M}$: $\widetilde{\mathbb{X}}(L/K)^{\times}\to\widetilde{\mathbb{X}}(M/K)^{\times}$ introduced in (2.17) and (2.18) possesses the following properties.

Lemma 2.21. *For the norm map* $\widetilde{\mathcal{N}}_{L/M}:\widetilde{\mathbb{X}}(L/K)^{\times}\to\widetilde{\mathbb{X}}(M/K)^{\times}$ *introduced by* (2.17) *and* (2.18)*, we have*

(i) $\widetilde{\mathcal{N}}_{L/M}\left(Z_{L/L_0}\left(\{E_i, f_i^{(L/L_o)}\}\right)\right)\subseteq Z_{M/M_0}\left(\{E_i\cap M, f_i^{(M/M_o)}\}\right)$;

(ii) $\widetilde{\mathcal{N}}_{L/M}\circ\langle\varphi\rangle_{L/M}\left(Y_{L/L_0}\right)\subseteq Y_{M/M_0}$.

Proof. (i) Recall that $\widetilde{\mathcal{N}}_{L/M}:\widetilde{\mathbb{X}}(L/K)^{\times}\to\widetilde{\mathbb{X}}(M/K)^{\times}$ is a continuous mapping. Now, for any choice of $z^{(i)}\in\mathrm{im}(f_i^{(L/L_o)})$, the continuity of the multiplicative arrow $\widetilde{\mathcal{N}}_{L/M}:\widetilde{\mathbb{X}}(L/K)^{\times}\to\widetilde{\mathbb{X}}(M/K)^{\times}$ yields

$$\widetilde{\mathcal{N}}_{L/M}\left(\prod_i z^{(i)}\right)=\prod_i\widetilde{\mathcal{N}}_{L/M}(z^{(i)}),$$

where $\widetilde{\mathcal{N}}_{L/M}(z^{(i)})\in\mathrm{im}(f_i^{(M/M_o)})$ by the commutative square (2.46).

(ii) For $y\in Y_{L/L_0}$, since $y^{1-\varphi^d}\in Z_{L/L_0}$, it follows that $\widetilde{\mathcal{N}}_{L/M}(y^{1-\varphi^d})\in Z_{M/M_0}$ by part (i). Observe that

$$\widetilde{\mathcal{N}}_{L/M}(y^{1-\varphi^d})=\widetilde{\mathcal{N}}_{L/M}(y)^{1-\varphi^d}=\left(\widetilde{\mathcal{N}}_{L/M}(y)^{1+\varphi^{d'}+\cdots\varphi^{d'(f(L/M)-1)}}\right)^{1-\varphi^{d'}}.$$

Therefore,

$$\widetilde{\mathcal{N}}_{L/M}(y)^{1+\varphi^{d'}+\cdots\varphi^{d'(f(L/M)-1)}}=\widetilde{\mathcal{N}}_{L/M}\circ\langle\varphi\rangle_{L/M}(y)\in Y_{M/M_0},$$

as desired. □

Part (ii) of Lemma 2.21 shows that the homomorphism $\widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M} : \widetilde{\mathbb{X}}(L/K)^{\times} \to \widetilde{\mathbb{X}}(M/K)^{\times}$ induces a group homomorphism, which will again be called the *Coleman norm map from L to M*; this homomorphism

$$\widetilde{\mathcal{N}}_{L/M}^{\text{Coleman}} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \to U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/M_0} \tag{2.47}$$

is defined by the formula

$$\widetilde{\mathcal{N}}_{L/M}^{\text{Coleman}}\left(\overline{U}\right) = \widetilde{\mathcal{N}}_{L/M} \circ \langle\varphi\rangle_{L/M}(U).Y_{M/M_0} \tag{2.48}$$

for every $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}$, where, as usual, $\overline{U}$ denotes the coset $U.Y_{L/L_0}$ in $U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$.

The following lemma is a refinement of Lemma 2.10.

Lemma 2.22. *Let K be a local field satisfying condition* (2.34). *For an infinite Galois subextension M/K of L/K such that the residue-class degree $[\kappa_M : \kappa_K]$ is equal to d' and $K \subset M \subset K_{\varphi^{d'}}$ for some $d' \mid d$, the following square is commutative*:

$$\begin{array}{ccc} \mathrm{Gal}(L/L_0) & \xrightarrow{\Phi^{(\varphi^d)}_{L/L_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0)}/Y_{L/L_0} \\ {\scriptstyle \mathrm{res}_M}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\ \mathrm{Gal}(M/M_0) & \xrightarrow{\Phi^{(\varphi^{d'})}_{M/M_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/M_0)}/Y_{M/M_0}, \end{array} \tag{2.49}$$

where the right vertical arrow is the Coleman norm map $\widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}$ from L to M defined by (2.47) *and* (2.48).

Proof. It suffices to prove that the square

$$\begin{array}{ccc} U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} & \xrightarrow{c_{L/L_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \\ {\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\ U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)} & \xrightarrow{c_{M/M_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/M_0} \end{array}$$

is commutative, which is obvious. Then, uniting this square with the square (2.24), we obtain

$$\begin{array}{ccccc} \mathrm{Gal}(L/K) & \xrightarrow{\phi^{(\varphi^d)}_{L/L_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/U_{\mathbb{X}(L/K)} & \xrightarrow{c_{L/L_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/K} \\ {\scriptstyle \mathrm{res}_M}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}^{\text{Coleman}}_{L/M}} \\ \mathrm{Gal}(M/K) & \xrightarrow{\phi^{(\varphi^{d'})}_{M/M_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/U_{\mathbb{X}(M/K)} & \xrightarrow{c_{M/M_0}} & U^{\diamond}_{\widetilde{\mathbb{X}}(M/K)}/Y_{M/K} \end{array}$$

and the commutativity of the square (2.49) follows. □

Thus, we have the following theorem, which is a refinement of Theorem 2.11.

Theorem 2.23. *Let K be a local field satisfying condition* (2.34)*. For an infinite Galois subextension M/K of L/K such that the residue-class degree $[\kappa_M : \kappa_K]$ is equal to d' and $K \subset M \subset K_{\varphi^{d'}}$ for some $d' \mid d$, the following square is commutative*:

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \xrightarrow{\boldsymbol{\Phi}^{(\varphi)}_{L/K}} & K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \\ \Big\downarrow{\scriptstyle \mathrm{res}_M} & & \Big\downarrow{\scriptstyle \left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)} \\ \mathrm{Gal}(M/K) & \xrightarrow{\boldsymbol{\Phi}^{(\varphi)}_{M/K}} & K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(M/K)}/Y_{M/M_0}, \end{array}$$

where the right-vertical arrow

$$K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} \xrightarrow{\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)} K^\times/N_{M_0/K}M_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(M/K)}/Y_{M/M_0}$$

is defined by

$$\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right) : (\overline{a}, \overline{U}) \mapsto \left(e^{\mathrm{CFT}}_{L_0/M_0}(\overline{a}), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\overline{U})\right)$$

for every $(\overline{a}, \overline{U}) \in K^\times/N_{L_0/K}L_0^\times \times U^{\diamond}_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0}$. Here

$$e^{CFT}_{L_0/M_0} : K^\times/N_{L_0/K}L_0^\times \to K^\times/N_{M_0/K}M_0^\times$$

is the natural inclusion defined via the existence theorem of the local class field theory.

Proof. By the isomorphism defined by (2.1) and (2.2), for $\sigma \in \mathrm{Gal}(L/K)$ there exists a unique $0 \leq m \in \mathbb{Z}$ such that $\sigma \mid_{L_0} = \varphi^m$ and $\varphi^{-m}\sigma \in \mathrm{Gal}(L/L_0)$. Now, by definition,

$$\boldsymbol{\Phi}^{(\varphi)}_{L/K}(\sigma) = \left(\pi_K^m N_{L_0/K}L_0^\times, \Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right).$$

Thus,

$$\begin{aligned} &\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)\left(\pi_K^m N_{L_0/K}L_0^\times, \Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma)\right) \\ &= \left(e^{\mathrm{CFT}}_{L_0/M_0}(\pi_K^m N_{L_0/K}L_0^\times), \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}(\Phi^{(\varphi^d)}_{L/L_0}(\varphi^{-m}\sigma))\right) \\ &= \left(\pi_K^m N_{M_0/K}M_0^\times, \Phi^{(\varphi^{d'})}_{M/M_0}(\varphi^{-m}\sigma \mid_M)\right) \end{aligned}$$

by Lemma 2.22. Note that, by the existence theorem of local class field theory, we have

$$e^{\mathrm{CFT}}_{L_0/M_0}(\pi_K^m N_{L_0/K}L_0^\times) = \pi_K^m N_{M_0/K}M_0^\times = \pi_K^{m'} N_{M_0/K}M_0^\times,$$

where $0 \le m' \in \mathbb{Z}$ is a unique integer satisfying $(\sigma\mid_M)\mid_{M_0} = \sigma\mid_{M_0} = \varphi^{m'}$ and $\varphi^{-m'}(\sigma\mid_M) \in \mathrm{Gal}(M/M_0)$. Hence,

$$\left(e^{\mathrm{CFT}}_{L_0/M_0}, \widetilde{\mathcal{N}}^{\mathrm{Coleman}}_{L/M}\right)(\mathbf{\Phi}^{(\varphi)}_{L/K}(\sigma)) = \left(\pi_K^{m'} N_{M_0/K}M_0^\times, \Phi^{(\varphi^{d'})}_{M/M_0}(\varphi^{-m}\sigma\mid_M)\right)$$
$$= \left(\pi_K^{m'} N_{M_0/K}M_0^\times, \Phi^{(\varphi^{d'})}_{M/M_0}(\varphi^{-m'}(\sigma\mid_M))\right) = \mathbf{\Phi}^{(\varphi)}_{M/K}(\mathrm{res}_M(\sigma))$$

by Remark 2.2 part (i), which completes the proof. □

Let K be a local field satisfying condition (2.34). Let F/K be a finite subextension of L/K. Thus, L/F is an infinite APF-Galois extension (see [8, Lemma 3.3]), where F satisfies (2.34). Fix a Lubin–Tate splitting φ_F over F and assume that the residue-class degree $[\kappa_L : \kappa_F]$ is equal to d' for some $d' \mid d$ and that there exists a chain of field extensions

$$F \subset L \subset F_{(\varphi_F)^{d'}}.$$

Then we have the generalized Fesenko reciprocity map

$$\mathbf{\Phi}^{(\varphi_F)}_{L/F} : \mathrm{Gal}(L/F) \to F^\times/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/L_0^{(F)}}$$

corresponding to the extension L/F. Here, as usual, $L_0^{(F)}$ is defined by $L_0^{(F)} = L \cap F^{nr} = F^{nr}_{d'}$ (we recall that $L_0^{(K)} = L \cap K^{nr} = K^{nr}_d$).

Now, we fix a basic sequence

$$L_0^{(K)} = E_o \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset L$$

for the extension $L/L_0^{(K)}$. Using the notation of [3] and [8], for each $1 \le i \in \mathbb{Z}$ we introduce an element σ_i in $\mathrm{Gal}(\widetilde{L}/\widetilde{K})$ such that $\langle \sigma\mid_{E_i}\rangle = \mathrm{Gal}(E_i/E_{i-1})$. Next, we fix the sequence

$$L_0^{(F)} = E_oL_0^{(F)} \subseteq E_1L_0^{(F)} \subseteq \cdots \subseteq E_iL_0^{(F)} \subseteq \cdots \subseteq LL_0^{(F)} = L$$

for the extension, $L/L_0^{(F)}$, as in [8, (5.55)]. For $1 \le i \in \mathbb{Z}$, we introduce elements σ_i^* in $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$ that satisfy

$$< \sigma_i^* \mid_{E_iL_0^{(F)}} >= \mathrm{Gal}(E_iL_0^{(F)}/E_{i-1}L_0^{(F)})$$

as follows :

(i) for $i > i_o$ we put $\sigma_i^* = \sigma_i$;

(ii) for $i \le i_o$ we put

$$\sigma_i^* = \begin{cases} \sigma_i & \text{if } E_{i-1}L_0^{(F)} \subset E_i L_0^{(F)}, \\ \mathrm{id}_{E_i L_0^{(F)}} & \text{if } E_{i-1}L_0^{(F)} = E_i L_0^{(F)}, \end{cases}$$

where i_o is defined as in [8, (5.55)]. Then it is clear that, for each $1 \le i \in \mathbb{Z}$, the elements σ_i^* of $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$ satisfy

$$< \sigma_i^* \mid_{E_i L_0^{(F)}} >= \mathrm{Gal}(E_i L_0^{(F)}/E_{i-1}L_o^{(F)}),$$

and that $\sigma_i^* = \sigma_i$ for almost all i. Next, for all $1 \le k, i \in \mathbb{Z}$, we introduce the map $h_k^{(L/L_0^{(F)})} : \prod_{1\le i\le k} U^{\sigma_i^*-1}_{\underbrace{}_{E_k L_0^{(F)}}} \to \Big(\prod_{1 \le i \le k+1} U^{\sigma_i^*-1}_{\underbrace{}_{E_{k+1}L_0^{(F)}}}\Big)/U^{\sigma_{k+1}^*-1}_{\underbrace{}_{E_{k+1}L_0^{(F)}}}$, the map $g_k^{(L/L_0^{(F)})} : \prod_{1\le i\le k} U^{\sigma_i^*-1}_{\underbrace{}_{E_k L_0^{(F)}}} \to \prod_{1\le i\le k+1} U^{\sigma_i^*-1}_{\underbrace{}_{E_{k+1}L_0^{(F)}}}$ and the map $f_i^{(L/L_0^{(F)})}$: $U^{\sigma_i^*-1}_{\underbrace{}_{E_i L_0^{(F)}}} \to U_{\widetilde{\mathbb{X}}(L/E_i L_0^{(F)})} \xrightarrow{\Lambda_{E_i L_0^{(F)}/L_0^{(F)}}} U_{\widetilde{\mathbb{X}}(L/F)}$ as in [3] and [8]. For each $1 \le k \in \mathbb{Z}$, we define a homomorphism

$$h_k^{(L/L_0^{(K)})} : \prod_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{E}_k} \to \left(\prod_{1\le i\le k+1} U^{\sigma_i-1}_{\widetilde{E}_{k+1}}\right)/U^{\sigma_{k+1}-1}_{\widetilde{E}_{k+1}}$$

that satisfies

$$\widetilde{N}^*_{E_{k+1}L_0^{(F)}/E_{k+1}} \circ h_k^{(L/L_0^{(F)})} = h_k^{(L/L_0^{(K)})} \circ \widetilde{N}_{E_k L_0^{(F)}/E_k}$$

and take any map

$$g_k^{(L/L_0^{(K)})} : \prod_{1\le i\le k} U^{\sigma_i-1}_{\widetilde{E}_k} \to \prod_{1\le i\le k+1} U^{\sigma_i-1}_{\widetilde{E}_{k+1}}$$

that satisfies

$$\widetilde{N}_{E_{k+1}L_0^{(F)}/E_{k+1}} \circ g_k^{(L/L_0^{(F)})} = g_k^{(L/L_0^{(K)})} \circ \widetilde{N}_{E_k L_0^{(F)}/E_k},$$

again following the same lines of [3] and [8].

Now, for each $1 \le i \in \mathbb{Z}$, we introduce the map $f_i^{(L/L_0^{(K)})} : U^{\sigma_i-1}_{\widetilde{E}_i} \to U_{\widetilde{\mathbb{X}}(L/K)}$ by $f_i^{(L/L_0^{(K)})}(w) = \Lambda_{F/K}\big(f_i^{(L/L_0^{(F)})}(v)\big)$, where $v \in U^{\sigma_i^*-1}_{\underbrace{}_{E_i L_0^{(F)}}}$ is any element satisfying $\widetilde{N}_{E_i L_0^{(F)}/E_i}(v) = w \in U^{\sigma_i-1}_{\widetilde{E}_i}$. Note that if $v' \in U^{\sigma_i^*-1}_{\underbrace{}_{E_i L_0^{(F)}}}$ is such that $\widetilde{N}_{E_i L_0^{(F)}/E_i}(v') = w$, then $\Lambda_{F/K}\big(f_i^{(L/L_0^{(F)})}(v)\big) = \Lambda_{F/K}\big(f_i^{(L/L_0^{(F)})}(v')\big)$.

Indeed, there exists $u \in \ker\big(\widetilde{N}_{E_i L_0^{(F)}/E_i}\big)$ such that $v' = vu$. Thus, we need to verify that $\Lambda_{F/K}\big(f_i^{(L/L_0^{(F)})}(v)\big) = \Lambda_{F/K}\big(f_i^{(L/L_0^{(F)})}(vu)\big)$. That is, for each $1 \leq j \in \mathbb{Z}$, we need to check the relation

$$\mathrm{Pr}_{\widetilde{E}_j}\left(\Lambda_{F/K}(f_i^{(L/L_0^{(F)})}(v))\right) = \mathrm{Pr}_{\widetilde{E}_j}\left(\Lambda_{F/K}(f_i^{(L/L_0^{(F)})}(vu))\right). \tag{2.50}$$

For $j > i$, we have

$$\begin{aligned}
\mathrm{Pr}_{\widetilde{E}_j}\left(\Lambda_{F/K}(f_i^{(L/L_0^{(F)})}(v))\right) &= \widetilde{N}_{E_j L_0^{(F)}/E_j}\left(\mathrm{Pr}_{\widetilde{E_j L_0^{(F)}}}\left(\Lambda_{F/K}(f_i^{(L/L_0^{(F)})}(v))\right)\right) \\
&= \widetilde{N}_{E_j L_0^{(F)}/E_j}\left(\mathrm{Pr}_{\widetilde{E_j L_0^{(F)}}}\left(f_i^{(L/L_0^{(F)})}(v)\right)\right) \\
&= \widetilde{N}_{E_j L_0^{(F)}/E_j}\left(g_{j-1}^{(L/L_0^{(F)})} \circ \cdots \circ g_i^{(L/L_0^{(F)})}(v)\right) \\
&= g_{j-1}^{(L/L_0^{(K)})} \circ \cdots \circ g_i^{(L/L_0^{(K)})}\left(\widetilde{N}_{E_i L_0^{(F)}/E_i}(v)\right),
\end{aligned}$$

by the properties of the mappings $g_k^{(L/L_0^{(F)})}$ and $g_k^{(L/L_0^{(K)})}$. Thus, relation (2.50) follows, because $\widetilde{N}_{E_i L_0^{(F)}/E_i}(v) = \widetilde{N}_{E_i L_0^{(F)}/E_i}(vu)$. Therefore, the map

$$f_i^{(L/L_0^{(K)})} : U_{\widetilde{E}_i}^{\sigma_i - 1} \to U_{\widetilde{\mathbb{X}}(L/K)}$$

is well defined. Moreover, for $j > i$, we have

$$\mathrm{Pr}_{\widetilde{E}_j} \circ f_i^{(L/L_0^{(K)})} = \left(g_{j-1}^{(L/L_0^{(K)})} \circ \cdots \circ g_i^{(L/L_0^{(K)})}\right)\big|_{U_{\widetilde{E}_i}^{\sigma_i - 1}}.$$

Indeed, for $w \in U_{\widetilde{E}_i}^{\sigma_i - 1}$, there exists $v \in U^{\sigma_i^* - 1}_{\widetilde{E_i L_0^{(F)}}}$ such that $\widetilde{N}_{E_i L_0^{(F)}/E_i}(v) = w$, and $f_i^{(L/L_0^{(K)})}(w) = \Lambda_{F/K}\big(f_i^{(L/L_0^{(F)})}(v)\big)$. That is, the square

$$\begin{array}{ccc}
U^{\sigma_i^* - 1}_{\widetilde{E_i L_0^{(F)}}} & \xrightarrow{f_i^{(L/L_0^{(F)})}} & U_{\widetilde{\mathbb{X}}(L/F)} \\
{\scriptstyle \widetilde{N}_{E_i L_0^{(F)}/E_i}} \big\downarrow & & \big\downarrow {\scriptstyle \Lambda_{F/K}} \\
U_{\widetilde{E}_i}^{\sigma_i - 1} & \xrightarrow{f_i^{(L/L_0^{(K)})}} & U_{\widetilde{\mathbb{X}}(L/K)}
\end{array} \tag{2.51}$$

is commutative. Thus,

$$\begin{aligned}\mathrm{Pr}_{\widetilde{E}_j} \circ f_i^{(L/L_0^{(K)})}(w) &= \mathrm{Pr}_{\widetilde{E}_j} \circ \Lambda_{F/K}\left(f_i^{(L/L_0^{(F)})}(v)\right)\\ &= \widetilde{N}_{E_jL_0^{(F)}/E_j}\left(\mathrm{Pr}_{\widetilde{E_jL_0^{(F)}}} \circ f_i^{(L/L_0^{(F)})}(v)\right)\\ &= \widetilde{N}_{E_jL_0^{(F)}/E_j}\left((g_{j-1}^{(L/L_0^{(F)})} \circ \cdots \circ g_i^{(L/L_0^{(F)})})(v)\right)\\ &= \left(g_{j-1}^{(L/L_0^{(K)})} \circ \cdots \circ g_i^{(L/L_0^{(K)})}\right)\left(\widetilde{N}_{E_iL_0^{(F)}/E_i}(v)\right),\end{aligned}$$

by the properties of the mappings $g_k^{(L/L_0^{(F)})}$ and $g_k^{(L/L_0^{(K)})}$, as claimed.

Now, we modify Lemma 5.30 of [8] and show that the homomorphism $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/L_0^{(F)})^\times \to \widetilde{\mathbb{X}}(L/L_0^{(K)})^\times$ introduced in (2.25) and (2.26) possesses the following properties.

Lemma 2.24. *For the continuous homomorphism* $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/L_0^{(F)})^\times \to \widetilde{\mathbb{X}}(L/L_0^{(K)})^\times$ *introduced by* (2.25) *and* (2.26) *we have*

(i) $\Lambda_{F/K}\big(Z_{L/L_0^{(F)}}(\{K_iF, f_i^{(L/L_0^{(F)})}\})\big) \subseteq Z_{L/L_0^{(K)}}\big(\{K_i, f_i^{(L/L_0^{(K)})}\}\big)$;

(ii) $\Lambda_{F/K}(Y_{L/L_0^{(F)}}) \subseteq Y_{L/L_0^{(K)}}$.

Proof. (i) For any choice of $z^{(i)} \in Z_i^{(L/L_0^{(F)})}$, the continuity of the multiplicative arrow $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/L_0^{(F)})^\times \to \widetilde{\mathbb{X}}(L/L_0^{(K)})^\times$ yields

$$\Lambda_{F/K}\left(\prod_i z^{(i)}\right) = \prod_i \Lambda_{F/K}(z^{(i)}),$$

where $\Lambda_{F/K}(z^{(i)}) \in Z_i^{(L/L_0^{(K)})}$ by the commutative square (2.51).

(ii) Let $y \in Y_{L/L_0^{(F)}}$. Then $y^{1-\varphi_F^{d'}} \in Z_{L/L_0^{(F)}}\big(\{K_iF, f_i^{(L/L_0^{(F)})}\}\big)$. Thus, $\Lambda_{F/K}(y^{1-\varphi_F^{d'}}) = \Lambda_{F/K}(y)^{1-\varphi_F^{d'}} \in Z_{L/L_0^{(K)}}\big(\{K_i, f_i^{(L/L_0^{(K)})}\}\big)$ by part (i). Now the result follows, because $\varphi_F^{d'} = \varphi_K^d$ by Remark 2.12. □

Thus, the homomorphism $\Lambda_{F/K} : \widetilde{\mathbb{X}}(L/L_0^{(F)})^\times \to \widetilde{\mathbb{X}}(L/L_0^{(K)})^\times$ defined by (2.26) induces a group homomorphism

$$\lambda_{F/K} : U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}} \to U^\diamond_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/Y_{L/L_0^{(K)}} \tag{2.52}$$

defined by

$$\lambda_{F/K}(\overline{U}) = \Lambda_{F/K}(U).Y_{L/L_0^{(K)}} \tag{2.53}$$

for every $U \in U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}$, where, as usual, $\overline{U}$ denotes the coset $U.Y_{L/L_0^{(F)}}$ in $U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}}$.

Let

$$\mathbf{\Phi}^{(\varphi_F)}_{L/F} : \mathrm{Gal}(L/F) \to F^{\times}/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}}$$

be the corresponding generalized Fesenko reciprocity map defined for the extension L/F, where $Y_{L/L_0^{(F)}} = Y_{L/L_0^{(F)}}\big(\{K_iF, f_i^{(L/L_0^{(F)})}\}\big)$.

The following lemma is a refinement of Lemma 2.13.

Lemma 2.25. *Let K be a local field satisfying condition* (2.34)*. Let F/K be a finite subextension of L/K. Fix a Lubin–Tate splitting φ_F over F and assume that the residue-class degree $[\kappa_L : \kappa_F]$ is equal to d' and $F \subset L \subset F_{(\varphi_F)^{d'}}$ for some $d' \mid d$. Then the square*

$$\begin{array}{ccc} \mathrm{Gal}(L/L_0^{(F)}) & \xrightarrow{\Phi^{(\varphi_K^d)}_{L/L_0^{(F)}}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}} \\ \text{inc.}\big\downarrow & & \big\downarrow \lambda_{F/K} \\ \mathrm{Gal}(L/L_0^{(K)}) & \xrightarrow{\Phi^{(\varphi_K^d)}_{L/L_0^{(K)}}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/Y_{L/L_0^{(K)}}, \end{array} \tag{2.54}$$

where the right vertical arrow $\lambda_{F/K} : U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}} \to U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/Y_{L/L_0^{(K)}}$ is defined by (2.52) *and* (2.53)*, is commutative.*

Proof. It suffices to prove that the square

$$\begin{array}{ccc} U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/U_{\mathbb{X}(L/L_0^{(F)})} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}} \\ \lambda_{F/K}\big\downarrow & & \big\downarrow \lambda_{F/K} \\ U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/U_{\mathbb{X}(L/L_0^{(K)})} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/Y_{L/L_0^{(K)}} \end{array}$$

is commutative, which is obvious. Then, uniting this square with the square (2.29), we obtain

$$\begin{array}{ccccc} \mathrm{Gal}(L/L_0^{(F)}) & \xrightarrow{\Phi^{(\varphi_K^d)}_{L/L_0^{(F)}}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/U_{\mathbb{X}(L/L_0^{(F)})} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(F)})}/Y_{L/L_0^{(F)}} \\ \Big\downarrow{\scriptstyle \text{inc.}} & & \Big\downarrow{\scriptstyle \lambda_{L/F}} & & \Big\downarrow{\scriptstyle \lambda_{L/F}} \\ \mathrm{Gal}(L/L_0^{(K)}) & \xrightarrow{\Phi^{(\varphi_K^d)}_{L/L_0^{(K)}}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/U_{\mathbb{X}(L/L_0^{(K)})} & \xrightarrow{\text{can.}} & U^{\diamond}_{\widetilde{\mathbb{X}}(L/L_0^{(K)})}/Y_{L/L_0^{(K)}}, \end{array}$$

and the commutativity of the square (2.54) follows. □

Thus, we have the following theorem, which is a refinement of Theorem 2.14.

Theorem 2.26. *Let K be a local field satisfying condition* (2.34)*. Let F/K be a finite subextension of L/K. Fix a Lubin–Tate splitting φ_F over F and assume that the residue-class degree $[\kappa_L : \kappa_F]$ is equal to d' and $F \subset L \subset F_{(\varphi_F)^{d'}}$ for some $d' \mid d$. Then the following square is commutative*:

$$\begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\mathbf{\Phi}^{(\varphi_F)}_{L/F}} & F^{\times}/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/L_0^{(F)}} \\ \Big\downarrow{\scriptstyle \text{inc.}} & & \Big\downarrow{\scriptstyle (N_{F/K},\lambda_{F/K})} \\ \mathrm{Gal}(L/K) & \xrightarrow{\mathbf{\Phi}^{(\varphi_K)}_{L/K}} & K^{\times}/N_{L_0^{(K)}/K}L_0^{(K)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0^{(K)}}, \end{array} \tag{2.55}$$

where the right vertical arrow

$$\begin{aligned} (N_{F/K},\lambda_{F/K}) : F^{\times}/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/L_0^{(F)}} \\ \to K^{\times}/N_{L_0^{(K)}/K}L_0^{(K)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/K)}/Y_{L/L_0^{(K)}} \end{aligned}$$

is defined by $(N_{F/K},\lambda_{F/K}) : (\overline{a},\overline{U}) \mapsto \big(\overline{N_{F/K}(a)},\lambda_{F/K}(\overline{U})\big)$ for every $(\overline{a},\overline{U}) \in F^{\times}/N_{L_0^{(F)}/F}L_0^{(F)\times} \times U^{\diamond}_{\widetilde{\mathbb{X}}(L/F)}/Y_{L/L_0^{(F)}}$.

Proof. Let $\sigma \in \mathrm{Gal}(L/F)$. There exists $0 \leq m \in \mathbb{Z}$ such that $\sigma \mid_{L_0^{(F)}} = \varphi_F^m$ and $\varphi_F^{-m}\sigma \in \mathrm{Gal}(L/L_0^{(F)})$. We have

$$\mathbf{\Phi}^{(\varphi_F)}_{L/F}(\sigma) = \Big(\pi_F^m . N_{L_0^{(F)}/F}L_0^{(F)\times}, \Phi^{(\varphi_K^d)}_{L/L_0^{(F)}}(\varphi_F^{-m}\sigma)\Big)$$

and

$$(N_{F/K}, \lambda_{F/K})(\mathbf{\Phi}^{(\varphi_F)}_{L/F}(\sigma)) = \left(\pi_K^m . N_{L_0^{(K)}/K} L_0^{(K)\times}, \Phi^{(\varphi_K^d)}_{L/L_0^{(K)}}(\varphi_F^{-m}\sigma)\right)$$

by the norm-compatibility of primes in the fixed Lubin–Tate labeling and by Lemma 2.25. Now, there exists $0 \leq m' \in \mathbb{Z}$ such that $\sigma \mid_{L_0^{(K)}} = \varphi_K^{m'}$ and $\varphi_K^{-m'}\sigma \in \mathrm{Gal}(L/L_0^{(K)})$. By part (ii) of Remark 2.2, it follows that $\varphi_F^m \mid_{L_0^{(K)}} = \varphi_K^{m'}$ and $\varphi_F^{-m}\sigma = \varphi_K^{-m'}\sigma$. By the Abelian local class field theory, $N_{F/K} : \pi_F^m N_{L_0^{(F)}/F} L_0^{(F)\times} \mapsto \pi_K^{m'} N_{L_0^{(K)}/K} L_0^{(K)\times} = \pi_K^m . N_{L_0^{(K)}/K} L_0^{(K)\times}$. Thus,

$$\begin{aligned}(N_{F/K}, \lambda_{F/K})(\mathbf{\Phi}^{(\varphi_F)}_{L/F}(\sigma)) &= \left(\pi_K^m . N_{L_0^{(K)}/K} L_0^{(K)\times}, \Phi^{(\varphi_K^d)}_{L/L_0^{(K)}}(\varphi_F^{-m}\sigma)\right) \\ &= \left(\pi_K^{m'} . N_{L_0^{(K)}/K} L_0^{(K)\times}, \Phi^{(\varphi_K^d)}_{L/L_0^{(K)}}(\varphi_K^{-m'}\sigma)\right) = \mathbf{\Phi}^{(\varphi_K)}_{L/K}(\sigma),\end{aligned}$$

which completes the proof. □

Finally, the inverse $\boldsymbol{H}^{(\varphi)}_{L/K} = (\mathbf{\Phi}^{(\varphi)}_{L/K})^{-1}$ of the generalized Fesenko reciprocity map $\mathbf{\Phi}^{(\varphi)}_{L/K}$ defined for the extension L/K is a generalization of the Hazewinkel map for infinite APF-Galois subextensions L/K of K_{φ^d}/K satisfying $[\kappa_L : \kappa_K] = d$ and under the assumption that the local field K satisfies condition (2.34). More precisely, the following is true.

Proposition 2.27. *The square*

$$\begin{array}{ccc} K^\times/N_{L_0/K}L_0^\times \times U^\diamond_{\tilde{\mathbb{X}}(L/K)}/Y_{L/L_0} & \xrightarrow{\boldsymbol{H}^{(\varphi)}_{L/K}} & \mathrm{Gal}(L/K) \\ \big\downarrow {\scriptstyle (id_{K^\times/N_{L_0/K}L_0^\times}, Pr_{\tilde{K}})} & & \big\downarrow {\scriptstyle \mathrm{mod}\ \mathrm{Gal}(L/K)'} \\ K^\times/N_{L_0/K}L_0^\times \times U_{L_0}/N_{L/L_0}U_L & \xrightarrow{h_{L/K}} & \mathrm{Gal}(L/K)^{ab} \end{array}$$

is commutative, where $h_{L/K} : K^\times/N_{L_0/K}L_0^\times \times U_{L_0}/N_{L/L_0}U_L \to \mathrm{Gal}(L/K)^{ab}$ *is the Hazewinkel map of* L/K.

References

[1] Fesenko I. B, *Local reciprocity cycles*, Invitation to Higher Local Fields (Münster, 1999) (I. B. Fesenko, M. Kurihara, eds.), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 293–298.

[2] Fesenko I. B., *Nonabelian local reciprocity maps*, Class Field Theory — Its Centenary and Prospect (Tokyo, 1998) (K. Miyake, ed.), Adv. Stud. Pure Math., vol. 30, Math. Soc. Japan, Tokyo, 2001, pp. 63–78.

[3] Fesenko I. B., *On the image of noncommutative local reciprocity map*, Homology, Homotopy Appl. **7** (2005), 53–62.
[4] Fesenko I. B., Vostokov S. V., *Local fields and their extensions. A constructive approach*, Transl. Math. Monogr., vol. 121, Amer. Math. Soc., Providence, RI, 1993.
[5] Fontaine J.-M., Wintenberger J.-P., *Le "corps des normes" de certaines extensions algébriques de corps locaux*, C. R. Acad. Sci. Paris Sér. A Math. **288** (1979), 367–370.
[6] Fontaine J.-M., Wintenberger J.-P., *Extensions algébriques et corps des normes des extensions APF des corps locaux*, C. R. Acad. Sci. Paris Sér. A Math. **288** (1979), 441–444.
[7] Gurevich A., *Ph.D. Thesis*, Humboldt Univ., Berlin, 1997.
[8] Ikeda K. I., Serbest E., *Fesenko reciprocity map*, Алгебра и анализ **20** (2008), №3.
[9] Ikeda K. I., Serbest E., *Non-abelian local class field theory*, Preprint, 2007.
[10] Koch H., de Shalit E., *Metabelian local class field theory*, J. Reine Angew. Math. **478** (1996), 85–106.
[11] Laubie F., *Une théorie du corps de classes local non abélien*, Compositio Math. **143** (2007), 339–362.
[12] Wintenberger J.-P., *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4) **46** (1983), 59–89.

Department of Mathematics
Istanbul Bilgi University
Kurtuluş Deresi Cad. No. 47, Dolapdere
34440 Beyoğlu, Istanbul
Turkey
E-mail: ilhan@bilgi.edu.tr

Поступило 20 октября 2007 г.

Gümüş Pala Mahallesi
Gümüş Sok., Öz Aksu Sitesi
C-2/39, 34160 Avcılar, Istanbul
Turkey
E-mail: erols73@yahoo.com