ON THE ε-FACTORS OF WEIL-DELIGNE REPRESENTATIONS

KAZIM İLHAN İKEDA*

ABSTRACT. An explicit expression for the ε-factor $\varepsilon_K((V,N),\psi,d\mu)$ of a representation (V,N) of the Weil-Deligne group WD_K of a local field K is given in terms of the non-abelian local class field theory of K.

CONTENTS

1. INTRODUCTION

2000 *Mathematics Subject Classification.* 11F80, 11S37.

Key words and phrases. Local Galois representations, Weil-Deligne representations, Local constants, non-abelian local class field theory.

*Member of "Feza Gürsey Fizik ve Matematik Araştırma ve Uygulama Merkezi", Kandilli, Istanbul.

1.1. **Aim of this work.** Let $\psi : K^+ \to \mathbb{C}^\times$ be a non-trivial additive character of K, and $d\mu$ an additive Haar measure on K^+. For a continuous complex representation V of the Weil group W_K of a local field K, let $\varepsilon_K^{\text{Deligne}}(V,\psi,d\mu)$ denote Deligne's local constant of the representation V of W_K with respect to ψ and $d\mu$ (look at [1] and [15]). In particular, choosing the Haar measure $d\mu_\psi$ on K^+ self-dual with respect to the additive character ψ of K^+, Langlands' local constant $\varepsilon_K^{\text{Langlands}}(V,\psi)$ of the representation V of W_K with respect to ψ (look at [11] and [15]) is defined by

$$\varepsilon_K^{\text{Langlands}}(V,\psi) = \varepsilon_K^{\text{Deligne}}(V \otimes \omega_{1/2}, \psi, d\mu_\psi),$$

where $\omega_{1/2} : W_K \to \mathbb{C}^\times$ is defined by $\omega_{1/2}(w) = \mid \mathsf{Art}_K(w) \mid_K^{1/2}$ for every $w \in W_K$. Here, $\mathsf{Art}_K : W_K \to K^\times$ denotes the local Artin reciprocity map of K, which induces the local Artin reciprocity isomorphism $\mathsf{Art}_{K*} : W_K^{\text{ab}} \xrightarrow{\sim} K^\times$ of K. Moreover, introduce

$$\varepsilon_K^{\text{Langlands}}(s,V,\psi) = \varepsilon_K^{\text{Deligne}}(V \otimes \omega_s, \psi, d\mu_\psi)$$

for $s \in \mathbb{C}$.

Note that, ε-factors are very important and central in the theory of Artin (and more generally motivic) L-functions. In fact, if $\underline{K}$ denotes a global field, the "completed" L-function $L_{\underline{K}}(s,V)$ of a representation $W_{\underline{K}} \to \mathrm{GL}(V)$ of the Weil group $W_{\underline{K}}$ of the global field $\underline{K}$ defined by the product

$$L_{\underline{K}}(s,V) = \prod_{v \in \mathbb{h}_{\underline{K}} \cup \mathbb{a}_{\underline{K}}} L_{\underline{K}_v}(s,V_v)$$

is convergent for $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) \gg \sigma$, for some σ, and defines a meromorphic function in the whole complex plane satisfying the functional equation

$$L_{\underline{K}}(s,V) = \varepsilon_{\underline{K}}(s,V) L_{\underline{K}}(1-s,V^*),$$

where V^* is the dual of the representation V of $W_{\underline{K}}$, and the ε-factor $\varepsilon_{\underline{K}}(s,V)$ of the representation V of $W_{\underline{K}}$ appearing mysteriously in the functional equation of $L_{\underline{K}}(s,V)$ and containing the basic information about the representation V of $W_{\underline{K}}$ over $\mathbb{C}$ is defined by the product

$$\varepsilon_{\underline{K}}(s,V) = \prod_{v \in \mathbb{h}_{\underline{K}} \cup \mathbb{a}_{\underline{K}}} \varepsilon_{\underline{K}_v}^{\text{Langlands}}(s,V_v,\psi_v),$$

where the product does not depend on the choice of a non-trivial additive character ψ of $\mathbb{A}_{\underline{K}}^+$ trivial on $\underline{K}$, where the local component of ψ at a place v is denoted by ψ_v, which is a non-trivial additive character of the local field $\underline{K}_v^+$.

The *root number* $W_{\underline{K}}(V)$ of the representation V of the Weil group $W_{\underline{K}}$ of the global field $\underline{K}$ is defined by the quotient

$$W_{\underline{K}}(V) = \frac{\varepsilon_{\underline{K}}(\frac{1}{2},V)}{\mid \varepsilon_{\underline{K}}(\frac{1}{2},V) \mid}.$$

Setting $W_{\underline{K}_v}(V_v,\psi_v) = \frac{\varepsilon_{\underline{K}_v}^{\text{Langlands}}(V,\psi_v)}{|\varepsilon_{\underline{K}_v}^{\text{Langlands}}(V,\psi_v)|}$ for each $v \in \mathbb{h}_{\underline{K}} \cup \mathbb{a}_{\underline{K}}$, the global root number $W_{\underline{K}}(V)$ of V is the product of local root numbers $W_{\underline{K}_v}(V_v,\psi_v)$ of V_v with respect to ψ_v as

$$W_{\underline{K}}(V) = \prod_{v \in \mathbb{h}_{\underline{K}} \cup \mathbb{a}_{\underline{K}}} W_{\underline{K}_v}(V_v,\psi_v).$$

Root numbers and ε-factors are very important and interesting objects, for example a weaker version of Birch and Swinnerton-Dyer Conjecture, which is called the Parity Conjecture claims that, for any elliptic curve E over $\mathbb{Q}$, the congruence

$$\mathrm{ord}_{s=1}L(s,E) \equiv \mathrm{rank}E(\mathbb{Q}) \pmod 2$$

should hold, where the parity of the analytic rank is connected with the root number of E; that is, the sign of the functional equation of $L(s,E)$. Thus, it is important to have an explicit expression or a formula for ε-factors.

By Deligne, Dwork, and Langlands' fundamental theorem (look at [1, 11, 15]), if $\dim(V) = 1$, then there is an explicit expression of the local ε-factor $\varepsilon_K^{\text{Langlands}}(V,\psi)$ of V with respect to ψ in terms of the local Artin reciprocity isomorphism $\mathsf{Art}_{K*} : W_K^{ab} \xrightarrow{\sim} K^\times$ of the local field K. On the other hand, if $\dim(V) \neq 1$, there is unfortunately *no known* explicit formula for the ε-factor [1] $\varepsilon_K^{\text{Langlands}}(V,\psi)$ of V with respect to ψ or $\varepsilon_K^{\text{Deligne}}(V,\psi,d\mu)$ of V with respect to ψ and $d\mu$.

Following Remark 5.6 of [4], the first aim of this paper, is to get an explicit expression for the local ε-factor $\varepsilon_K^{\text{Langlands}}(V,\psi)$ of a continuous semisimple representation V of W_K with respect to ψ in terms of the non-abelian local class field theory developed in [7, 8]; or equivalently, by the Laubie's theory [12] (also look at [6]). The second aim is to get explicit expressions for the local ε-factor $\varepsilon_K^{\text{Langlands}}((V,N),\psi)$ of a Weil-Deligne representation (V,N) of the Weil-Deligne group WD_K of K with respect to ψ and for the local ε-factor $\varepsilon_K^{\text{Langlands}}(\varphi,\psi)$ of an admissible representation φ of the Weil-Arthur group WA_K of K with respect to ψ.

The paper is organized as follows. Parts 2 and 3, 4 are preliminary in nature. More precisely, following [1, 11, 15], in Part 2, we shall recall Deligne, Dwork, and Langlands' fundamental theorem on the existence and uniqueness of local ε-factors of continuous representations of W_K and their basic properties. In Part 3, first we shall briefly review Weil-Deligne representations of the group WD_K over a field E of characteristic 0, and then introduce the local ε-factors of Weil-Deligne representations of WD_K over $\mathbb{C}$. Next, in Part 4, we shall first recall admissible representations of Weil-Arthur group WA_K of K and, after discussing the relationship between Weil-Deligne representations of WD_K over $\mathbb{C}$ and admissible representations of WA_K over $\mathbb{C}$, define ε-factors of admissible representations of WA_K over $\mathbb{C}$. Parts 5 and 6, 7, occupy the major portion of the paper, where we provide an explicit description of the local ε-factor $\varepsilon_K^{\text{Langlands}}(V,\psi)$ of a continuous representation V of G_K with respect to a fixed non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K in terms of the non-abelian local class field theory developed in the papers [7, 8, 6, 12]. The main idea is to introduce an integral expression $\mathrm{I}_K(V,\psi)$ similar to the integral expression $\int_{U_K} \chi_*(\mathsf{Art}_K^{-1}(y))^{-1}\psi(y/c)d\mu_\psi(y)$ appearing in the formula for the abelian local constant

[1] The lack of such an explicit formula has been pointed out by Tate, in part (3.4) of [15], as follows:

> "... For this there is at present only an existence theorem (see below), no explicit formula. This lack is not surprising if we recall that the formulas defining ε in (3.2) make essential use of the interpretation of χ as a quasi-character of F^*; if we think of χ as a quasi-character of W_F, we have no way to define $\varepsilon(\chi,\psi,dx)$ without using the reciprocity law isomorphism $F^* \approx W_F^{\text{ab}}$. In fact it was his idea about "nonabelian reciprocity laws" relating representations of degree n of W_F to irreducible representations π of $\mathrm{GL}(n,F)$, and the possibility of defining $\varepsilon(\pi,\psi,dx)$ for the latter, which led Langlands to conjecture and prove a version of the following big ..."

$\varepsilon_K^{\text{Tate}}(\chi,\psi)\in\mathbb{C}^\times$ of Tate given by

$$\varepsilon_K^{\text{Tate}}(\chi,\psi)=\chi_*(\mathsf{Art}_K^{-1}(c))\frac{\int_{U_K}\chi_*(\mathsf{Art}_K^{-1}(y))^{-1}\psi(y/c)d\mu_\psi(y)}{|\int_{U_K}\chi_*(\mathsf{Art}_K^{-1}(y))^{-1}\psi(y/c)d\mu_\psi(y)|},$$

but which is modeled after the non-abelian local class field theory, and then apply Deligne, Dwork, and Langlands' theorem to this setting. Finally, in Part 8, which is the last part of this work, we shall provide a formula for ε-factors of local Langlands parameters

$$WA_K\to{}^L\mathrm{G}$$

of a reductive group G defined over K, which is necessary in our future research.

1.2. **Notation.** All through this work K denotes a local field with finite residue-class field $\kappa_K=O_K/\mathfrak{p}_K$ of $q_K=:q=p^f$ elements, where O_K denotes the ring of integers in K with the unique prime ideal $\mathfrak{p}_K$. As usual, the unit group of K is denoted by U_K and the higher unit groups of K by U_K^i, where $0\le i\in\mathbb{Z}$. The normalized discrete valuation on K is denoted by $v_K:K\to\mathbb{Z}\cup\{\infty\}$ and the corresponding absolute value on K is denoted by $|\bullet|_K\colon K\to\mathbb{R}_{\ge0}$. In this text, we fix a separable closure K^{sep} of K once and for all. The absolute Galois group of K and the absolute Weil group of K are denoted by G_K and W_K respectively. The group G_K is equipped with its Krull topology, and W_K is topologized by declaring the inertia subgroup I_K of W_K to be open, while I_K is equipped with the profinite topology. Thus, I_K is a compact group. The higher ramification subgroups of G_K in upper numbering are denoted by G_K^v, where $-1\le v\in\mathbb{R}$. Note that, $G_K^0=I_K$, the inertia group of K. The higher ramification subgroups W_K^v of W_K are defined by $W_K^v=W_K\cap G_K^v$, for $-1\le v\in\mathbb{R}$. Then, W_K^0 is the inertia subgroup I_K of W_K. So, it follows that $W_K^v=G_K^v$ for every $0\le v\in\mathbb{Z}$. For a finite separable extension L of the local field K, the equality $W_K^v\cap W_L=W_L^{\psi_{L/K}(v)}$ holds for $v\ge0$, where $\psi_{L/K}:\mathbb{R}_{\ge-1}\to\mathbb{R}_{\ge-1}$ is the Hasse-Herbrand function of the extension L/K. We shall denote the local Artin reciprocity map of K by

$$\mathsf{Art}_K:W_K\to K^\times,$$

and the local Artin reciprocity isomorphism of K by

$$\mathsf{Art}_{K*}:W_K^{ab}\xrightarrow{\sim}K^\times.$$

Finally, let WD_K denote the Weil-Deligne group of K.

The collection of all isomorphism classes of continuous complex semisimple representations of W_K is denoted by $\mathrm{M}(W_K)$, and the group of all virtual representations of W_K by $\mathrm{R}(W_K)$. As usual, $\mathrm{R}^0(W_K)$ denotes the subgroup of $\mathrm{R}(W_K)$ consisting of all virtual representations of W_K of degree 0.

2. Deligne–Dwork–Langlands' Theorem

In this part, we shall first recall the Theorem of Deligne, Dwork, and Langlands about the existence and uniqueness of ε-factors of continuous representations of the Weil group W_K of K, and then list their basic properties. The main reference for this part is [15].

2.1. **Existence and uniqueness of ε-factors of continuous representations of the Weil group W_K of K over $\mathbb{C}$.** A fundamental result of Deligne, Dwork, and Langlands states, for each local field K, the existence of a *unique* assignment

$$\varepsilon_K^{\text{Langlands}}:(V,\psi)\rightsquigarrow\varepsilon_K^{\text{Langlands}}(V,\psi)$$

which associates with each pair (V,ψ) consisting of a continuous semisimple representation V of W_K over $\mathbb{C}$ and a non-trivial additive character $\psi:K^+\to\mathbb{C}^\times$ of K, a non-zero number $\varepsilon_K^{\text{Langlands}}(V,\psi)\in\mathbb{C}^\times$, called Langlands' local ε-factor (i.e. Langlands' local constant) of the pair (V,ψ), such that:

(i) If V and V' are equivalent representations of W_K over $\mathbb{C}$, then

$$\varepsilon_K^{\text{Langlands}}(V,\psi)=\varepsilon_K^{\text{Langlands}}(V',\psi);$$

(ii) If V is a 1-dimensional continuous complex representation of W_K, and $\chi: W_K\to\mathbb{C}^\times$ is the corresponding quasi-character, then the equality

$$\varepsilon_K^{\text{Langlands}}(V,\psi)=\varepsilon_K^{\text{Tate}}(\chi,\psi)$$

is satisfied. Here, the number $\varepsilon_K^{\text{Tate}}(\chi,\psi)\in\mathbb{C}^\times$ is the abelian local constant of Tate, which is defined by

$$\varepsilon_K^{\text{Tate}}(\chi,\psi)=\chi_*(\mathsf{Art}_K^{-1}(c))\frac{\int_{U_K}\chi_*(\mathsf{Art}_K^{-1}(y))^{-1}\psi(y/c)d\mu_\psi(y)}{\mid\int_{U_K}\chi_*(\mathsf{Art}_K^{-1}(y))^{-1}\psi(y/c)d\mu_\psi(y)\mid},$$

where $\chi_*: W_K^{ab}\to\mathbb{C}^\times$ is the quasi-character of W_K^{ab} corresponding to the quasi-character χ of W_K, $d\mu_\psi$ is the additive Haar measure on K^+ self-dual with respect to the additive character ψ of K^+, and $c\in K^\times$ such that $v_K(c)=a(\chi)+n(\psi)$. Here, $a(\chi)$ denotes the exponent of the Artin conductor $\mathfrak{f}^{\text{Artin}}(\chi)$ of χ and $n(\psi)$ the largest integer n such that $\psi(\pi^{-n}O_K)=1$;

(iii) If E is any finite separable extension of K, then the function

$$\varepsilon_E^{\text{Langlands}}:(V,\psi\circ\text{Tr}_{E/K})\rightsquigarrow\varepsilon_E^{\text{Langlands}}(V,\psi\circ\text{Tr}_{E/K})$$

which associates with each pair $(V,\psi\circ\text{Tr}_{E/K})$ consisting of a semisimple representation V of W_E over $\mathbb{C}$ and an additive character $\psi: K^+\to\mathbb{C}^\times$ of K, a non-zero number $\varepsilon_E^{\text{Langlands}}(V,\psi\circ\text{Tr}_{E/K})\in\mathbb{C}^\times$ is "inductive in degree 0 over K". That is,

(a) The map

$$\varepsilon_E^{\text{Langlands}}(-,\psi\circ\text{Tr}_{E/K}):\mathrm{M}(W_E)\to\mathbb{C}^\times$$

is "additive". Namely, it satisfies

$$\varepsilon_E^{\text{Langlands}}(V,\psi\circ\text{Tr}_{E/K})=\varepsilon_E^{\text{Langlands}}(V',\psi\circ\text{Tr}_{E/K})\varepsilon_E^{\text{Langlands}}(V'',\psi\circ\text{Tr}_{E/K}),$$

for each short exact sequence

$$0\to V'\to V\to V''\to 0$$

of semisimple representations of W_E over $\mathbb{C}$;

(b) For every tower of finite separable extensions $K\subseteq E\subseteq E'$,

$$\varepsilon_E^{\text{Langlands}}(\text{Ind}_{W_{E'}}^{W_E}V,\psi\circ\text{Tr}_{E/K})=\varepsilon_{E'}^{\text{Langlands}}(V,\psi\circ\text{Tr}_{E'/K})$$

for every $V\in\mathrm{R}^0(W_{E'})$.

2.2. **Basic properties of $\varepsilon_K(V,\psi,d\mu)$.** Let $\psi: K^+\to\mathbb{C}^\times$ be a fixed non-trivial additive character of K, and $d\mu_\psi$ the additive Haar measure on K^+ self-dual with respect to the additive character ψ of K^+. Then (look at [15]):

(i) $\varepsilon_K^{\text{Langlands}}(V,\psi_a)=(\det V)(a)\varepsilon_K^{\text{Langlands}}(V,\psi)$, for every $a\in K^\times$. Here, for $a\in K^\times$, the additive character $\psi_a: K^+\to\mathbb{C}^\times$ of K is defined by $\psi_a(y)=\psi(ay)$ for every $y\in K^+$;

(ii) $\varepsilon_K^{\text{Langlands}}(V\otimes\omega_s,\psi)=q_K^{-s(n(\psi)\dim(V)+a(V))}\varepsilon_K^{\text{Langlands}}(V,\psi)$, where q_K is the number of elements of the finite residue-class field κ_K of K. For $s\in\mathbb{C}$, $\omega_s: W_K\to\mathbb{C}^\times$ is the quasi-character defined by $\omega_s(w)=\mid\text{Art}_K(w)\mid_K^s$ for every $w\in W_K$;

(iii) $\varepsilon_K^{\text{Deligne}}(V,\psi,d\mu) = (\frac{d\mu}{d\mu_\psi})^{\dim V}\varepsilon_K^{\text{Langlands}}(V\otimes\omega_{-\frac{1}{2}},\psi)$, for any additive Haar measure $d\mu$ on K^+.

For $s\in\mathbb{C}$, define

$$\varepsilon_K^{\text{Langlands}}(s,V,\psi) = \varepsilon_K^{\text{Deligne}}(V\otimes\omega_s,\psi,d\mu_\psi)$$

for $s\in\mathbb{C}$. Thus, the identity $\varepsilon_K^{\text{Langlands}}(s,V,\psi) = \varepsilon_K^{\text{Langlands}}(V\otimes\omega_{s-\frac{1}{2}},\psi)$ holds for $s\in\mathbb{C}$.

3. Weil-Deligne representations of WD_K over $\mathbb{C}$ and their ε-factors

In this part, we shall review the ε-factors of Weil-Deligne representations of WD_K over E, where E is a field of characteristic 0.

3.1. **Weil-Deligne representations of WD_K over E.**

Recall that, an n-dimensional Weil-Deligne representation of WD_K over E is a pair (V,N) consisting of:

(i) a continuous representation $\rho: W_K\to \mathrm{GL}(V)$ of W_K on an n-dimensional linear space V over E, where $\mathrm{GL}(V)$ is equipped with the discrete topology;

(ii) a nilpotent E-endomorphism N of V, such that

$$\rho(w)N\rho(w)^{-1} = \|\, w\,\|_K N,$$

for every $w\in W_K$. Here, $\|\, w\,\|_K = |\,\mathsf{Art}_K(w)\,|_K = q_K^{v_K(\mathsf{Art}_K(w))} = q_K^{\mathrm{v}_K(w)}$ for every $w\in W_K$.

Moreover, a Weil-Deligne representation (V,N) of WD_K over E is called Φ-semisimple, if $\rho: W_K\to\mathrm{GL}(V)$ is a semisimple representation of W_K on V over E in the usual sense.

Given any two Weil-Deligne representations (V,N) and (V',N') of WD_K over E, an intertwining operator $f: V\to V'$ satisfying $f\circ N = N'\circ f$ is called a morphism of Weil-Deligne representations from (V,N) to (V',N'). If the intertwining operator $f: V\to V'$ satisfying $f\circ N = N'\circ f$ is an equivalence of representations V and V', then (V,N) and (V',N') are said to be isomorphic or equivalent. Therefore, we can define the category of Weil-Deligne representations of WD_K over E, with $\oplus$, $\otimes$, and the dual given by

$$(V,N)\oplus(V',N') = (V\oplus V', \begin{pmatrix} N & 0\\ 0 & N'\end{pmatrix}),\ (V,N)\otimes(V',N') = (V\otimes V', N\otimes 1 + 1\otimes N'),$$

and

$$(V,N)^\vee = (V^\vee, -N^\vee)$$

respectively, for any Weil-Deligne representations (V,N) and (V',N') of WD_K over E.

For (V,N) any Weil-Deligne representation of WD_K over E, let $(V,N)_{\mathrm{ss}}$ denote the Φ-semisimplification of the representation (V,N) of WD_K over E, which is defined, following Tate [15], by

$$(V,N)_{\mathrm{ss}} = (Vu^{-\mathrm{v}_K}, N),$$

where $u\in\mathrm{End}(V)$ is the unique unipotent automorphism of V commuting with N and W_K and such that

$$\exp(aN)wu^{-\mathrm{v}_K(w)}$$

is a semisimple automorphism of V for every $a\in E$ and every $w\in W_K - I_K$. Note that (V,N) is Φ-semisimple if and only if $(V,N) = (V,N)_{\mathrm{ss}}$.

A Φ-semisimple Weil-Deligne representation (V,N) of WD_K over E is:

– Irreducible if and only if V is an irreducible representation of W_K over E and $N=0$;

– Indecomposable if and only if (V,N) is a "Steinberg representation"; that is, (V,N) has the form $(V_o,N_o)\otimes\mathrm{Sp}(d)$, where $\mathrm{Sp}(d)$ denotes the special representation of WD_K over E for some $1\leq d\in\mathbb{Z}$, and (V_o,N_o) denotes an irreducible Weil-Deligne representation of WD_K over E (so V_o is an irreducible representation of W_K over E and $N_o=0$).

Moreover, any Φ-semisimple Weil-Deligne representation (V,N) of WD_K over E is a finite direct sum

$$(V,N)=((V_1,N_1)\otimes\mathrm{Sp}(d_1))\oplus\cdots\oplus((V_r,N_r)\otimes\mathrm{Sp}(d_r))$$

of indecomposable Weil-Deligne representations $(V_j,N_j)\otimes\mathrm{Sp}(d_j)$ of WD_K over E, where V_j is an irreducible representation of W_K over E, $N_j=0$, and $\mathrm{Sp}(d_j)$ is the special representation of WD_K over E for some $1\leq d_j\in\mathbb{Z}$, for each $j=1,\cdots,r$ for some $1\leq r\in\mathbb{Z}$.

3.2. **ε-factors of Weil-Deligne representations over $\mathbb{C}$.** Recall that, Langlands' local ε-factor $\varepsilon^{\mathrm{Langlands}}((V,N),\psi)$ of a Weil-Deligne representation (V,N) of WD_K over $E\subseteq\mathbb{C}$ with respect to a non-trivial additive character $\psi:K^+\to\mathbb{C}^\times$ is defined by

$$\varepsilon^{\mathrm{Langlands}}((V,N),\psi)=\varepsilon^{\mathrm{Langlands}}(V,\psi).\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}}). \tag{3.1}$$

Here, V_N denotes the kernel $\ker(N)$ of the monodromy operator N on V and

$$V_N^{I_K}=\{v\in V_N\mid\rho(w)(v)=v,{}^\forall w\in I_K\}.$$

Note that, as a function of (V,N), $\varepsilon^{\mathrm{Langlands}}((V,N),\psi)$ is *not* additive. Moreover, define

$$\varepsilon^{\mathrm{Langlands}}((V,N),\psi;t)=\varepsilon^{\mathrm{Langlands}}((V,N),\psi).t^{a((V,N))}, \tag{3.2}$$

and

$$\begin{aligned}\varepsilon^{\mathrm{Langlands}}(s,(V,N),\psi)&=\varepsilon^{\mathrm{Langlands}}((V,N),\psi;t)\mid_{t=q^{-s}}\\&=\varepsilon^{\mathrm{Langlands}}((V,N),\psi).q^{-s.a((V,N))},\end{aligned} \tag{3.3}$$

for $s\in\mathbb{C}$, where the exponent $a((V,N))$ of the Artin conductor $\mathfrak{f}^{\mathrm{Artin}}((V,N))$ of the representation (V,N) of WD_K is defined by

$$a((V,N))=a(V)+\dim(V^{I_K})-\dim(V_N^{I_K}).$$

Recall that, the exponent $a(V)$ of the the Artin conductor $\mathfrak{f}^{\mathrm{Artin}}(V)$ of the representation V of W_K over E has an explicit expression (also look at [5]) in terms of higher ramification subgroups W_K^s of W_K as

$$a(V)=\int_{-1}^{\infty}\mathrm{codim}(V^{W_K^s})ds.$$

It is well known that

$$\varepsilon^{\mathrm{Langlands}}((V,N),\psi)=\varepsilon^{\mathrm{Langlands}}((V,N)_{\mathrm{ss}},\psi) \tag{3.4}$$

and

$$a((V,N))=a((V,N)_{\mathrm{ss}}).$$

So, assume that (V,N) is a Φ-semisimple representation of WD_K over $E\subseteq\mathbb{C}$. Then, V is a semisimple representation of W_K in the usual sense. Let

$$V=V_1\oplus\cdots\oplus V_k$$

be the decomposition of the representation V of W_K into irreducible representations of W_K. Then, by the additivity of $\varepsilon^{\mathrm{Langlands}}(V,\psi)$ as a function of V,

$$\varepsilon^{\mathrm{Langlands}}(V,\psi)=\varepsilon^{\mathrm{Langlands}}(V_1,\psi)\cdots\varepsilon^{\mathrm{Langlands}}(V_k,\psi).$$

Recall that, an irreducible representation V of W_K has the form $V^{\mathrm{Gal}} \otimes \omega_s$, where V^{Gal} is a Galois type representation of W_K and, for some $s \in \mathbb{C}$, ω_s is the 1-dimensional representation of W_K defined by $\omega_s(w) = \|w\|_K^s$, for every $w \in W_K$. Therefore, Langlands' local ε-factor $\varepsilon^{\mathrm{Langlands}}(V,\psi)$ of an irreducible representation V of W_K with respect to ψ has the form

$$\begin{aligned} \varepsilon^{\mathrm{Langlands}}(V,\psi) &= \varepsilon^{\mathrm{Langlands}}(V^{\mathrm{Gal}} \otimes \omega_s, \psi) \\ &= \varepsilon^{\mathrm{Langlands}}(V^{\mathrm{Gal}},\psi) q_K^{-s.[a(V^{\mathrm{Gal}})+n(\psi)\dim(V^{\mathrm{Gal}})]}. \end{aligned} \tag{3.5}$$

From this identity, it clearly follows that, we have to study Langlands' local ε-factors of Galois type representations of W_K (i.e. continuous representations of G_K) with respect to ψ.

4. Admissible representations of WA_K over $\mathbb{C}$ and their ε-factors

For automorphic purposes, it is convenient to replace Φ-semisimple Weil-Deligne representations of WD_K over $\mathbb{C}$ with the admissible representations of the Weil-Arthur group $WA_K = W_K \times \mathrm{SL}(2,\mathbb{C})$ of K over $\mathbb{C}$.

4.1. **Admissible representations of Weil-Arthur group WA_K of K over $\mathbb{C}$.** Recall that, an n-dimensional representation

$$\varphi : WA_K \to \mathrm{GL}(n,\mathbb{C})$$

of WA_K over $\mathbb{C}$ is called admissible, if (i) φ is trivial on an open subgroup of the inertia group I_K of K; (ii) $\varphi(\Phi_K)$ is semisimple; and (iii) $\varphi|_{\mathrm{SL}(2,\mathbb{C})}$ is an algebraic homomorphism. Two n-dimensional admissible representations φ_1 and φ_2 of WA_K over $\mathbb{C}$ are equivalent if φ_1 and φ_2 are $\mathrm{GL}(n,\mathbb{C})$-conjugate.

Now, Φ-semisimple Weil-Deligne representations of WD_K over $\mathbb{C}$ correspond to the admissible representations of the Weil-Arthur group $WA_K = W_K \times \mathrm{SL}(2,\mathbb{C})$ of K over $\mathbb{C}$ via a bijection between equivalence classes of Φ-semisimple Weil-Deligne representations (V,N) of WD_K over $\mathbb{C}$ and $\mathrm{GL}(n,\mathbb{C})$-conjugacy classes of admissible representations of the Weil-Arthur group $WA_K = W_K \times \mathrm{SL}(2,\mathbb{C})$ of K over $\mathbb{C}$. One direction of this bijection is defined, for any admissible representation $\varphi : WA_K \to \mathrm{GL}(n,\mathbb{C})$ of WA_K over $\mathbb{C}$, by

$$\varphi \rightsquigarrow (\rho_\varphi, N_\varphi),$$

where $(\rho_\varphi, N_\varphi)$ is the Φ-semisimple Weil-Deligne representation of WD_K over $\mathbb{C}$ defined by $N_\varphi = d\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$, where $d\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$ is the derivative of the n-dimensional Lie group representation $\varphi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$ over $\mathbb{C}$, and $\rho_\varphi : W_K \to \mathrm{GL}(n,\mathbb{C})$ is defined by

$$\rho_\varphi(w) = \varphi(w)\varphi\left(\begin{pmatrix} q_K^{\mathrm{v}_K(w)/2} & 0 \\ 0 & q_K^{-\mathrm{v}_K(w)/2} \end{pmatrix}\right),$$

for every $w \in W_K$. For the reverse direction, for any Φ-semisimple Weil-Deligne representation (V,N) of WD_K over $\mathbb{C}$, define the assignment

$$(V,N) \rightsquigarrow \varphi_{(V,N)},$$

where $\varphi_{(V,N)} : WA_K \to \mathrm{GL}(V)$ is the admissible representation of WA_K over $\mathbb{C}$, which is defined by the unique $\mathfrak{sl}_2$-triple (e,f,h) determined by the nilpotent endomorphism N of

V coming from (V,N), by

$$\varphi_{(V,N)}(w) = \exp\left(\frac{-\mathrm{v}(w)}{2}\log q_K.h\right)\rho(w),$$

for each $w \in W_K$, and by the representation of $\mathrm{SL}(2)$ on V over $\mathbb{C}$ determined by (e,f,h). Moreover, under this bijective correspondence $(V,N) \rightsquigarrow \varphi_{(V,N)}$, if the Φ-semisimple Weil-Deligne representation (V,N) of WD_K over $\mathbb{C}$ is:

- Irreducible, that is V is an irreducible representation of W_K over $\mathbb{C}$ and $N=0$, then the corresponding admissible representation $\varphi_{(V,N)}$ of WA_K over $\mathbb{C}$ is $V\otimes 1$;
- Indecomposable, that is $(V,N) = (V_o,N_o)\otimes \mathrm{Sp}(d)$, where V_o is an irreducible representation of W_K over $\mathbb{C}$ and $N_o = 0$, and $\mathrm{Sp}(d)$ is the special representation of WD_K over $\mathbb{C}$, then the corresponding admissible representation $\varphi_{(V,N)}$ of WA_K over $\mathbb{C}$ is $V_o\otimes V^{(d)}$, where $V^{(d)}$ is the unique irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ of dimension d over $\mathbb{C}$

Thus, if (V,N) is any Φ-semisimple Weil-Deligne representation of WD_K over $\mathbb{C}$, the corresponding admissible representation $\varphi_{(V,N)}$ of WA_K over $\mathbb{C}$ is

$$\varphi_{(V,N)} = (V_1\otimes V^{(d_1)})\oplus\cdots\oplus(V_r\otimes V^{(d_r)}),$$

where

$$(V,N) = ((V_1,N_1)\otimes \mathrm{Sp}(d_1))\oplus\cdots\oplus((V_r,N_r)\otimes \mathrm{Sp}(d_r))$$

is the finite direct sum decomposition of (V,N) into indecomposable Weil-Deligne representations $(V_j,N_j)\otimes\mathrm{Sp}(d_j)$ of WD_K over $\mathbb{C}$, where V_j is an irreducible representation of W_K over $\mathbb{C}$, $N_j=0$, and $\mathrm{Sp}(d_j)$ is the special representation of WD_K over $\mathbb{C}$ corresponding to the unique irreducible representation $V^{(d_j)}$ of $\mathrm{SL}(2,\mathbb{C})$ of dimension d_j over $\mathbb{C}$ for some $1\le d_j\in\mathbb{Z}$, for each $j=1,\cdots,r$ for some $1\le r\in\mathbb{Z}$.

4.2. **ε-factors of admissible representations of WA_K over $\mathbb{C}$.** Let $\varphi : WA_K \to \mathrm{GL}(V)$ be an admissible representation of the Weil-Arthur group WA_K of K on an n-dimensional space V over $\mathbb{C}$. Then we set

$$\varepsilon^{\text{Langlands}}(\varphi,\psi) = \varepsilon^{\text{Langlands}}((\rho_\varphi,N_\varphi),\psi), \tag{4.1}$$

$$\varepsilon^{\text{Langlands}}(\varphi,\psi;t) = \varepsilon^{\text{Langlands}}((\rho_\varphi,N_\varphi),\psi;t), \tag{4.2}$$

and

$$\varepsilon^{\text{Langlands}}(s,\varphi,\psi) = \varepsilon^{\text{Langlands}}(s,(\rho_\varphi,N_\varphi),\psi) \tag{4.3}$$

for $s\in\mathbb{C}$, and further set

$$a(\varphi) = a((\rho_\varphi,N_\varphi)) \text{ and } \mathfrak{f}^{\text{Artin}}(\varphi) = \mathfrak{f}^{\text{Artin}}((\rho_\varphi,N_\varphi)).$$

5. ε-FACTORS ATTACHED TO THE REPRESENTATIONS OF LOCAL ABSOLUTE GALOIS GROUPS OVER $\mathbb{C}$

In this part, we shall describe Langlands' local ε-factor of a continuous representation of the absolute Galois group G_K of the local field K with respect to an additive character $\psi : K^+ \to \mathbb{C}^\times$ of K.

5.1. **The maximal d-abelian extension $K^{d-\mathrm{ab}}$ of K.** Let $K^{d-\mathrm{ab}}$ denote the maximal "d-abelian" extension of K inside K^{sep}. That is, $K^{d-\mathrm{ab}}$ is the d-abelian closure of K inside K^{sep} defined as the union of all solvable extensions of K inside K^{sep} of derived length at most d. Note that, $K^{d-\mathrm{ab}}$ is a Galois extension of K whose Galois group $\mathrm{Gal}(K^{d-\mathrm{ab}}/K)$ is the maximal d-abelian Hausdorff quotient (that is, the "d-abelianization") $G_K/G_K^{[d]}$ of the absolute Galois group G_K of K, where $G_K^{[d]}$ denotes the closure of the d-th derived subgroup of G_K. Note that, for $d \leq d'$, $K^{d-\mathrm{ab}} \subseteq K^{d'-\mathrm{ab}}$ and there exists a morphism

$$\mathrm{Res}^{K^{d'-\mathrm{ab}}}_{K^{d-\mathrm{ab}}} : \mathrm{Gal}(K^{d'-\mathrm{ab}}/K) \to \mathrm{Gal}(K^{d-\mathrm{ab}}/K)$$

defined by restriction to $K^{d-\mathrm{ab}}$.

5.2. **Representations of the local Galois group G_K of K over $\mathbb{C}$.** Given a continuous representation

$$\rho : G_K \to \mathrm{GL}(V)$$

of the absolute Galois group G_K of the local field K on an n-dimensional vector space V over $\mathbb{C}$. As $\mathrm{im}(\rho)$ is a finite subgroup of $\mathrm{GL}(V)$, there exists a finite Galois extension K_ρ/K satisfying $\mathrm{Gal}(K^{\mathrm{sep}}/K_\rho) = \ker(\rho)$ and a faithful representation

$$\rho_* : \mathrm{Gal}(K_\rho/K) \to \mathrm{GL}(V)$$

of $\mathrm{Gal}(K_\rho/K)$ on V over $\mathbb{C}$. The group $\mathrm{Gal}(K_\rho/K)$ is solvable. Denote the derived length of the group $\mathrm{Gal}(K_\rho/K)$ by $\partial_K(\rho)$. For any integer d satisfying $\partial_K(\rho) \leq d$, there exists a natural continuous homomorphism

$$\mathrm{Res}^{K^{d-\mathrm{ab}}}_{K_\rho} : \mathrm{Gal}(K^{d-\mathrm{ab}}/K) \to \mathrm{Gal}(K_\rho/K)$$

defined by restriction to K_ρ, which in return defines a representation

$$\rho^{d-\mathrm{ab}} = \rho_* \circ \mathrm{Res}^{K^{d-\mathrm{ab}}}_{K_\rho} : \mathrm{Gal}(K^{d-\mathrm{ab}}/K) \xrightarrow{\mathrm{Res}^{K^{d-\mathrm{ab}}}_{K_\rho}} \mathrm{Gal}(K_\rho/K) \xrightarrow{\rho_*} \mathrm{GL}(V)$$

of $\mathrm{Gal}(K^{d-\mathrm{ab}}/K)$ on V over $\mathbb{C}$. Finally, for $\partial_K(\rho) \leq d \leq d'$, the following triangle

(5.1)
$$\begin{array}{ccc} \mathrm{Gal}(K^{d'-\mathrm{ab}}/K) & & \\ \downarrow \mathrm{Res}^{K^{d'-\mathrm{ab}}}_{K^{d-\mathrm{ab}}} & \searrow^{\rho^{d'-\mathrm{ab}}} & \\ & & \mathrm{GL}(V) \\ & \nearrow_{\rho^{d-\mathrm{ab}}} & \\ \mathrm{Gal}(K^{d-\mathrm{ab}}/K) & & \end{array}$$

is commutative.

5.3. **Non-abelian local class field theory.** Following [7, 8] and [12] closely, fix an extension φ_K of the Frobenius automorphism Fr_K of K^{nr} to K^{sep}. Namely, fix a *Lubin-Tate splitting* φ_K over K. The *non-abelian local class field theory of K (in the sense of Koch)*, constructed in [7, 8, 12], establishes an algebraic and topological isomorphism

$$\mathbf{\Phi}_K^{(\varphi_K)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_K)}$$

between the absolute Galois group G_K of the local field K and a certain topological group $\nabla_K^{(\varphi_K)}$, which depends only on K and on the Lubin-Tate splitting φ_K over K. The construction of the topological group $\nabla_K^{(\varphi_K)}$ involves the theory of *APF*-extensions of K and the fields of norms construction of Fontaine and Wintenberger.

Moreover, the isomorphism $\mathbf{\Phi}_K^{(\varphi_K)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_K)}$ is called the *non-abelian local reciprocity map of K*, because it is "natural" in the sense that the non-abelian analogues of the abelian local class field theoretic properties, such as "existence", "functoriality", and a certain "ramification theoretic" property, are all satisfied (look at Section 8 of [6] together with [8] for detailed account). Following [4], denote the inverse $\mathbf{\Phi}_K^{(\varphi_K)^{-1}} : \nabla_K^{(\varphi_K)} \xrightarrow{\sim} G_K$ of the non-abelian local reciprocity isomorphism $\mathbf{\Phi}_K^{(\varphi_K)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_K)}$ by

$$\{\bullet, K\}_{\varphi_K} : \nabla_K^{(\varphi_K)} \xrightarrow{\sim} G_K,$$

and call it the *non-abelian local norm-residue symbol of K*.

The non-abelian local reciprocity map $\mathbf{\Phi}_K^{(\varphi_K)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_K)}$ of K induces an isomorphism

$$\mathbf{\Phi}_K^{(\varphi_K)^{d-\mathrm{ab}}} : G_K^{d-\mathrm{ab}} \xrightarrow{\sim} \left(\nabla_K^{(\varphi_K)}\right)^{d-\mathrm{ab}},$$

with inverse

$$\{\bullet, K\}_{\varphi_K}^{d-\mathrm{ab}} : \left(\nabla_K^{(\varphi_K)}\right)^{d-\mathrm{ab}} \xrightarrow{\sim} G_K^{d-\mathrm{ab}},$$

for each $d = 1, 2, 3, \cdots$. If $d = 1$, then $\mathbf{\Phi}_K^{(\varphi_K)^{1-\mathrm{ab}}} = \widehat{\mathsf{Art}}_{K_*} : G_K^{\mathrm{ab}} \xrightarrow{\sim} \left(\nabla_K^{(\varphi_K)}\right)^{\mathrm{ab}} = \widehat{K^\times}$ is the local Artin reciprocity isomorphism of K "in Galois form" with inverse $\{\bullet, K\}_{\varphi_K}^{1-\mathrm{ab}} = (\bullet, K) : \left(\nabla_K^{(\varphi_K)}\right)^{\mathrm{ab}} = \widehat{K^\times} \xrightarrow{\sim} G_K^{\mathrm{ab}}$, which is the norm-residue symbol of K. Here, $\widehat{K^\times}$ is the profinite completion of $K^\times$.

There exist a dense subgroup ${}_{\mathbb{Z}}\nabla_K^{(\varphi_K)}$ of $\nabla_K^{(\varphi_K)}$ such that

$$\mathbf{\Phi}_K^{(\varphi_K)}(W_K) = {}_{\mathbb{Z}}\nabla_K^{(\varphi_K)},$$

a subgroup ${}_1\nabla_K^{(\varphi_K)^{\underline{0}}}$ of ${}_{\mathbb{Z}}\nabla_K^{(\varphi_K)}$ satisfying the equality

$$\mathbf{\Phi}_K^{(\varphi_K)}\left(G_K^0\right) = \mathbf{\Phi}_K^{(\varphi_K)}\left(W_K^0\right) = {}_1\nabla_K^{(\varphi_K)^{\underline{0}}},$$

and subgroups $\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}}$ of ${}_{\mathbb{Z}}\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}$ satisfying the equality

$$\mathbf{\Phi}_K^{(\varphi_K)^{d-\mathrm{ab}}}\left(G_K^{d-\mathrm{ab}^0}\right) = \mathbf{\Phi}_K^{(\varphi_K)^{d-\mathrm{ab}}}\left(W_K^{d-\mathrm{ab}^0}\right) = \left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}},$$

for $d = 1, 2, 3, \cdots$. Note that, the natural continuous homomorphism $W_K^{d-\mathrm{ab}} \to G_K^{d-\mathrm{ab}}$ is an injection with dense image, and the higher ramification subgroups $W_K^{d-\mathrm{ab}^v}$ of $W_K^{d-\mathrm{ab}}$ are defined by $W_K^{d-\mathrm{ab}^v} = W_K^{d-\mathrm{ab}} \cap G_K^{d-\mathrm{ab}^v}$ for $-1 \leq v \in \mathbb{R}$. In particular, for $v = 0$, $W_K^{d-\mathrm{ab}^0} = G_K^{d-\mathrm{ab}^0}$, as $G_K^{d-\mathrm{ab}^0} = \left(G_K/G_K^{[d]}\right)^0 = I_K/G_K^{(d)}$ for $d = 1, 2, 3, \cdots$. Note that, if $d = 1$, then $\mathbf{\Phi}_K^{(\varphi_K)^{1-\mathrm{ab}}} = \mathsf{Art}_{K_*} : W_K^{\mathrm{ab}^0} \xrightarrow{\sim} \left({}_1\nabla_K^{(\varphi_K)^{\mathrm{ab}}}\right)^{\underline{0}} = U_K$.

The following two tables summarize the abelian and the non-abelian local class field theories of K :

Non-abelian local C.F.T. (φ_K fixed)	
G_K	$\nabla_K^{(\varphi_K)}$
W_K	$_{\mathbb{Z}}\nabla_K^{(\varphi_K)}$
W_K^0	$_1\nabla_K^{(\varphi_K)\underline{0}}$
$W_K^\delta,\ \delta \in (i-1,i]$	$_1\nabla_K^{(\varphi_K)\underline{i}}$

$\xrightarrow{\text{ab}}$

Abelian local class field theory	
G_K^{ab}	$\widehat{K^\times}$
W_K^{ab}	$K^\times$
$W_K^{\text{ab}\,0}$	U_K
$W_K^{\text{ab}\,\delta},\ \delta \in (i-1,i]$	U_K^i

By [10], if E/K is any finite Galois extension, then there exists a finite unramified extension E'/E such that E'/K is “compatible” with respect to the Lubin-Tate splitting φ_K over K (that is, E' sits in the tower $K \subseteq E' \subset K_{\varphi_K^{d'}}$, where $d' = f(E'/K)$) and the following diagram

(5.2)
$$\begin{array}{ccc}
G_{E'} & \xrightarrow[\sim]{\mathbf{\Phi}_{E'}^{(\varphi_{E'})}} & \nabla_{E'}^{(\varphi_{E'})} \\
\downarrow{\scriptstyle \text{inc.}} & & \downarrow \\
G_E & \xrightarrow[\sim]{\mathbf{\Phi}_K^{(\varphi_K)}} & \mathbf{\Phi}_K^{(\varphi_K)}(G_E) \\
\downarrow{\scriptstyle \text{inc.}} & & \downarrow \\
G_K & \xrightarrow[\mathbf{\Phi}_K^{(\varphi_K)}]{\sim} & \nabla_K^{(\varphi_K)}
\end{array} \qquad \nabla_{E'}^{(\varphi_{E'})} \xrightarrow{\mathscr{N}_{E'/K}^{\infty}} \nabla_K^{(\varphi_K)}$$

is commutative (look at the proof of the commutativity of the diagram (7.12) in [7]).

5.4. **The Haar measure on the compact group $\left(\nabla_K^{(\varphi_K)}\right)^{d-\text{ab}}$.** Let $d\mu^{G_K}$ denote the unique Haar measure on the absolute Galois group G_K of the local field K. Likewise, let $d\mu^{G_K^{d-\text{ab}}}$ denote the unique Haar measure on $G_K^{d-\text{ab}}$ for $d = 1,2,3,\cdots$. Clearly, the Haar measure $d\mu^{G_K}$ on G_K induces a Haar measure on the d-abelianization $G_K^{d-\text{ab}}$ of G_K, which is nothing but $d\mu^{G_K^{d-\text{ab}}}$ for $d = 1,2,3,\cdots$.

The non-abelian local reciprocity map $\mathbf{\Phi}_K^{(\varphi_K)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_K)}$ of K and the d-abelianized reciprocity isomorphism $\mathbf{\Phi}_K^{(\varphi_K)\,d-\text{ab}} : G_K^{d-\text{ab}} \xrightarrow{\sim} \left(\nabla_K^{(\varphi_K)}\right)^{d-\text{ab}}$ for each $d = 1,2,3,\cdots$ define a Haar measure $d\mu^{\nabla_K}$ on $\nabla_K^{(\varphi_K)}$ and a Haar measure $d\mu^{\nabla_K^{d-\text{ab}}}$ on $\left(\nabla_K^{(\varphi_K)}\right)^{d-\text{ab}}$ for each $d = 1,2,3,\cdots$, respectively.

Now, assume that E/K is a finite and separable extension. Then, the identity

$$(G_K : G_E)d\mu^{G_K} = d\mu^{G_E}$$

together with the commutative diagram (5.2) yields

$$(\nabla_K^{(\varphi_K)} : \mathscr{N}_E^{\infty})d\mu^{\nabla_K} = d\mu^{\nabla_E},$$

where $\mathscr{N}_E^{\infty} = \mathbf{\Phi}_K^{(\varphi_K)}(G_E)$.

5.5. **Main theorems.** Fix a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of the local field K.

Let

$$\rho : G_K \to \mathrm{GL}(V)$$

be an irreducible continuous representation of the absolute Galois group G_K of K on an n-dimensional vector space V over $\mathbb{C}$. Following Section 5.2, for any integer d satisfying $\partial_K(\rho) \leq d$, let

$$\rho^{d-\mathrm{ab}} : G_K^{d-\mathrm{ab}} \to \mathrm{GL}(V)$$

denote the canonical representation of $G_K^{d-\mathrm{ab}}$ on V defined by $\rho : G_K \to \mathrm{GL}(V)$. Let

$$\chi_{\rho^{d-\mathrm{ab}}} : G_K^{d-\mathrm{ab}} \xrightarrow{\rho^{d-\mathrm{ab}}} \mathrm{GL}(V) \xrightarrow{\mathrm{Tr}} \mathbb{C}$$

be the character of the representation $V_{\rho^{d-\mathrm{ab}}}$ of $G_K^{d-\mathrm{ab}}$, and

$$\det_{\rho^{d-\mathrm{ab}}} : G_K^{d-\mathrm{ab}} \xrightarrow{\rho^{d-\mathrm{ab}}} \mathrm{GL}(V) \xrightarrow{\det} \mathbb{C}^\times$$

be the determinant of the representation $V_{\rho^{d-\mathrm{ab}}}$ of $G_K^{d-\mathrm{ab}}$. Note that, the homomorphism $\det_{\rho^{d-\mathrm{ab}}} : G_K^{d-\mathrm{ab}} \to \mathbb{C}^\times$ factors through

(5.3)
$$\begin{array}{ccc} G_K^{d-\mathrm{ab}} & \xrightarrow{\det_{\rho^{d-\mathrm{ab}}}} & \mathbb{C}^\times \\ & \searrow^{\text{canonical map}} \quad \nearrow_{\det_{\rho^{d-\mathrm{ab}}*}} & \\ & (G_K^{d-\mathrm{ab}})^{\mathrm{ab}} & \end{array}$$

where $(G_K^{d-\mathrm{ab}})^{\mathrm{ab}} = G_K^{\mathrm{ab}}$.

For the representation V_ρ of G_K and for any integer d satisfying $\partial_K(\rho) \leq d$, introduce

$$\mathrm{I}_K(V_\rho, \psi, d, c) = \int_{\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^0} \det_{\rho^{d-\mathrm{ab}}}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1} \psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right) d\mu^{\nabla_K^{d-\mathrm{ab}}}(y)$$

and

$$\mathrm{D}_K(V_\rho, \psi, d, c) = \det_{\rho^{d-\mathrm{ab}}*}\left(\mathsf{Art}_K^{-1}(c)\right),$$

where $c = c(V_\rho, \psi) \in K^\times$ such that $\nu_K(c) = a(V_\rho) + n(\psi)$. Note that, the integral $\mathrm{I}_K(V_\rho, \psi, d, c)$ exists and is a non-zero complex number.

Proposition 5.1. *The expression* $\mathrm{D}_K(V_\rho, \psi, d, c)\frac{\mathrm{I}_K(V_\rho,\psi,d,c)}{|\mathrm{I}_K(V_\rho,\psi,d,c)|}$ *does not depend on the choice of* $d \in \mathbb{Z}$ *satisfying* $\partial_K(\rho) \leq d$ *and* $c \in K^\times$ *satisfying* $\nu_K(c) = a(V_\rho) + n(\psi)$.

Proof. Let $c, c' \in K^\times$ satisfying $\nu_K(c) = \nu_K(c') = a(V_\rho) + n(\psi)$. Let $u \in U_K$ so that $c' = u.c$. Then,

$$\begin{aligned} \mathrm{D}_K(V_\rho, \psi, d, c') &= \det_{\rho^{d-\mathrm{ab}}*}\left(\mathsf{Art}_K^{-1}(c')\right) \\ &= \det_{\rho^{d-\mathrm{ab}}*}\left(\mathsf{Art}_K^{-1}(u.c)\right) \\ &= \det_{\rho^{d-\mathrm{ab}}*}\left(\mathsf{Art}_K^{-1}(u)\right)\mathrm{D}_K(V_\rho, \psi, d, c), \end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_K(V_\rho,\psi,d,c') &= \int\limits_{\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}}} \det_{\rho^{d-\mathrm{ab}}}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c'^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right)d\mu^{\nabla_K^{d-\mathrm{ab}}}(y)\\
&= \int\limits_{\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}}} \det_{\rho^{d-\mathrm{ab}}}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}u^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right)d\mu^{\nabla_K^{d-\mathrm{ab}}}(y)\\
&= \int\limits_{\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}}} \det_{\rho^{d-\mathrm{ab}}}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{u^{-1}y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right)d\mu^{\nabla_K^{d-\mathrm{ab}}}(y)\\
&= \det_{\rho^{d-\mathrm{ab}}}\left(\{u,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\int\limits_{\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}}} \det_{\rho^{d-\mathrm{ab}}}\left(\{u^{-1}y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{u^{-1}y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right) d\\
&= \det_{\rho^{d-\mathrm{ab}}{}_*}\left(\mathsf{Art}_K^{-1}(u)\right)^{-1}\int\limits_{\left({}_1\nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}\right)^{\underline{0}}} \det_{\rho^{d-\mathrm{ab}}}\left(\{u^{-1}y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{u^{-1}y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right) d\\
&= \det_{\rho^{d-\mathrm{ab}}{}_*}\left(\mathsf{Art}_K^{-1}(u)\right)^{-1}\mathrm{I}_K(V_\rho,\psi,d,c),
\end{aligned}$$

where the last equality follows from the left-invariance of the Haar measure $d\mu^{\nabla_K^{d-\mathrm{ab}}}$. Thus,

$$\mathrm{D}_K(V_\rho,\psi,d,c')\frac{\mathrm{I}_K(V_\rho,\psi,d,c')}{|\,\mathrm{I}_K(V_\rho,\psi,d,c')\,|} = \mathrm{D}_K(V_\rho,\psi,d,c)\frac{\mathrm{I}_K(V_\rho,\psi,d,c)}{|\,\mathrm{I}_K(V_\rho,\psi,d,c)\,|},$$

as $\det_{\rho^{d-\mathrm{ab}}{}_*}\left(\mathsf{Art}_K^{-1}(u)\right)^{-1}\in\boldsymbol{\mu}_\infty(K^{\mathrm{alg}})$, where $\boldsymbol{\mu}_\infty(K^{\mathrm{alg}})$ denotes the multiplicative group of all roots of unity in a fixed algebraic closure K^{alg} of K.

Next, let $d,d'\in\mathbb{Z}$ such that $\partial_K(\rho)\le d\le d'$. Choose any $\sigma\in G_K$ such that, setting $\sigma^{d'-\mathrm{ab}} := \sigma\mid_{K^{d'-\mathrm{ab}}}$, the equality $\sigma^{d'-\mathrm{ab}}\mid_{K^{\mathrm{ab}}} = \mathsf{Art}_K^{-1}(c)$ holds. Then,

$$\det_{\rho^{d'-\mathrm{ab}}}\left(\sigma^{d'-\mathrm{ab}}\right) = \det_{\rho^{d'-\mathrm{ab}}{}_*}\left(\mathsf{Art}_K^{-1}(c)\right),$$

by the commutativity of the diagram (5.3). On the other hand, this $\sigma\in G_K$ also satisfies $\sigma^{d-\mathrm{ab}} := \sigma\mid_{K^{d-\mathrm{ab}}} = \sigma^{d'-\mathrm{ab}}\mid_{K^{d-\mathrm{ab}}}$ and $\sigma^{d-\mathrm{ab}}\mid_{K^{\mathrm{ab}}} = \mathsf{Art}_K^{-1}(c)$, and again by the commutative triangle (5.3),

$$\det_{\rho^{d-\mathrm{ab}}}\left(\sigma^{d-\mathrm{ab}}\right) = \det_{\rho^{d-\mathrm{ab}}{}_*}\left(\mathsf{Art}_K^{-1}(c)\right).$$

Thus, by the commutative diagram (5.1),

$$\det_{\rho^{d'-\mathrm{ab}}}\left(\sigma^{d'-\mathrm{ab}}\right) = \det_{\rho^{d-\mathrm{ab}}}\left(\sigma^{d-\mathrm{ab}}\right),$$

which proves that

$$\mathrm{D}_K(V_\rho,\psi,d',c) = \mathrm{D}_K(V_\rho,\psi,d,c).$$

Moreover, changing variables along the continuous (hence measurable) surjection

$$r_d^{d'} : \nabla_K^{(\varphi_K)^{d'-\mathrm{ab}}} \twoheadrightarrow \nabla_K^{(\varphi_K)^{d-\mathrm{ab}}}$$

which makes the diagram

$$\begin{array}{ccc} {\nabla_K^{(\varphi_K)}}^{d'-\mathrm{ab}} & \xrightarrow[\sim]{\{\bullet,K\}_{\varphi_K}^{d'-\mathrm{ab}}} & G_K^{d'-\mathrm{ab}} \\ {\scriptstyle r_d^{d'}}\downarrow & & \downarrow {\scriptstyle \mathrm{Res}_{K^{d-\mathrm{ab}}}^{K^{d'-\mathrm{ab}}}} \\ {\nabla_K^{(\varphi_K)}}^{d-\mathrm{ab}} & \xrightarrow[\sim]{\{\bullet,K\}_{\varphi_K}^{d-\mathrm{ab}}} & G_K^{d-\mathrm{ab}} \end{array} \tag{5.4}$$

commutative, the following equalities

$$\begin{aligned} \mathrm{I}_K(V_\rho,\psi,d,c) &= \int\limits_{\left({}_1{\nabla_K^{(\varphi_K)}}^{d-\mathrm{ab}}\right)^{\underline{0}}} \det{}_{\rho^{d-\mathrm{ab}}}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{y,K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right)d(r_d^{d'})_*\mu^{\nabla_K^{d'-\mathrm{ab}}}(y) \\ &= \int\limits_{\left({}_1{\nabla_K^{(\varphi_K)}}^{d'-\mathrm{ab}}\right)^{\underline{0}}} \det{}_{\rho^{d-\mathrm{ab}}}\left(\{r_d^{d'}(y),K\}_{\varphi_K}^{d-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\{r_d^{d'}(y),K\}_{\varphi_K}^{d-\mathrm{ab}}\right)\right)d\mu^{\nabla_K^{d'-\mathrm{ab}}}(y) \end{aligned}$$

hold, where the first equality follows from compactness of ${\nabla_K^{(\varphi_K)}}^{d-\mathrm{ab}}$ (so the push-forward measure $d(r_d^{d'})_*\mu^{\nabla_K^{d'-\mathrm{ab}}}$ on ${\nabla_K^{(\varphi_K)}}^{d-\mathrm{ab}}$ under $r_d^{d'}$ is nothing but $d\mu^{\nabla_K^{d'-\mathrm{ab}}}$) and the second equality is the change of variables formula along the surjection $r_d^{d'}$. Therefore, by commutative diagrams (5.4) and (5.1), and by the fact that

$$\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\mathrm{Res}_{K^{d-\mathrm{ab}}}^{K^{d'-\mathrm{ab}}}\{y,K\}_{\varphi_K}^{d'-\mathrm{ab}}\right) = \mathsf{Art}_{K^{d'-\mathrm{ab}}/K}\left(\{y,K\}_{\varphi_K}^{d'-\mathrm{ab}}\right)$$

for every $y \in {\nabla_K^{(\varphi_K)}}^{d'-\mathrm{ab}}$, the equalities

$$\begin{aligned} \mathrm{I}_K(V_\rho,\psi,d,c) &= \int\limits_{\left({}_1{\nabla_K^{(\varphi_K)}}^{d'-\mathrm{ab}}\right)^{\underline{0}}} \det{}_{\rho^{d-\mathrm{ab}}}\left(\mathrm{Res}_{K^{d-\mathrm{ab}}}^{K^{d'-\mathrm{ab}}}\{y,K\}_{\varphi_K}^{d'-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d-\mathrm{ab}}/K}\left(\mathrm{Res}_{K^{d-\mathrm{ab}}}^{K^{d'-\mathrm{ab}}}\{y,K\}_{\varphi_K}^{d'-\mathrm{ab}}\right)\right)d\mu^{\nabla_K^{d'-\mathrm{ab}}}(y) \\ &= \int\limits_{\left({}_1{\nabla_K^{(\varphi_K)}}^{d'-\mathrm{ab}}\right)^{\underline{0}}} \det{}_{\rho^{d'-\mathrm{ab}}}\left(\{y,K\}_{\varphi_K}^{d'-\mathrm{ab}}\right)^{-1}\psi\left(c^{-1}\mathsf{Art}_{K^{d'-\mathrm{ab}}/K}\left(\{y,K\}_{\varphi_K}^{d'-\mathrm{ab}}\right)\right)d\mu^{\nabla_K^{d'-\mathrm{ab}}}(y) \\ &= \mathrm{I}_K(V_\rho,\psi,d',c), \end{aligned}$$

follow and completes the proof. □

So, by Proposition 5.1, the constant $\mathrm{D}_K(V_\rho,\psi,d,c)\frac{\mathrm{I}_K(V_\rho,\psi,d,c)}{|\mathrm{I}_K(V_\rho,\psi,d,c)|}$ depends *only* on the irreducible representation V_ρ of G_K and on the non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K.

Introduce

$$\mathrm{e}_K(V,\psi) := \mathrm{D}_K(V,\psi,d,c)\frac{\mathrm{I}_K(V,\psi,d,c)}{|\,\mathrm{I}_K(V,\psi,d,c)\,|},$$

where V is any *irreducible* representation of G_K over $\mathbb{C}$ and ψ is a fixed non-trivial additive character of K. Moreover, for an *arbitrary* continuous representation $\rho : G_K \to \mathrm{GL}(V)$ of

G_K on an n-dimensional vector space V over $\mathbb{C}$, with the decomposition $V_\rho = V_{\rho_1} \oplus \cdots \oplus V_{\rho_m}$ into irreducible subrepresentations $V_{\rho_1}, \cdots, V_{\rho_m}$ of V_ρ, define

$$\mathrm{e}_K(V_\rho, \psi) := \prod_{j=1}^{m} \mathrm{e}_K(V_{\rho_j}, \psi).$$

Lemma 5.2. *Assume that $K \subseteq E \subseteq E' \subseteq E''$ is a tower of finite and separable extensions of K. Let*

$$\chi : G_{E''} \to \mathrm{GL}(\mathbb{C})$$

be a 1-dimensional complex representation of $G_{E''}$. Then

$$\mathrm{e}_{E'}(\mathrm{Ind}_{G_{E''}}^{G_{E'}} \chi, \psi \circ \mathrm{Tr}_{E'/K}) = \mathrm{e}_{E''}(\chi, \psi \circ \mathrm{Tr}_{E''/K}),$$

where $\psi : K^+ \to \mathbb{C}^\times$ is a fixed non-trivial additive character of K.

Assume that $K \subseteq E \subseteq E'$ is a tower of finite and separable extensions of K. Let

$$r_1', r_2' : G_{E'} \to \mathrm{GL}(V)$$

be two continuous representations of $G_{E'}$ on an n-dimensional space V over $\mathbb{C}$. By Brauer Induction Theorem, there are $\mathbb{Z}$-linear combinations

$$r_1' = \sum_{i=1}^{s} n_i \mathrm{Ind}_{G_{E_i'}}^{G_{E'}} \chi_i$$

and

$$r_2' = \sum_{i=1}^{s} m_i \mathrm{Ind}_{G_{E_i'}}^{G_{E'}} \tau_i,$$

where $n_i, m_i \in \mathbb{Z}$ (note that 0 is allowed in this setting), $G_{E_i'}$ are certain finite index subgroups of $G_{E'}$, and $\chi_i, \tau_i : G_{E_i'} \to \mathrm{GL}(\mathbb{C})$ are certain 1-dimensional complex representations of $G_{E_i'}$ for $i = 1, \cdots, s$. Then,

$$\mathrm{Ind}_{G_{E'}}^{G_E} r_1' = \mathrm{Ind}_{G_{E'}}^{G_E}(\sum_{i=1}^{s} n_i \mathrm{Ind}_{G_{E_i'}}^{G_{E'}} \chi_i) = \sum_{i=1}^{s} n_i \mathrm{Ind}_{G_{E_i'}}^{G_E} \chi_i$$

and

$$\mathrm{Ind}_{G_{E'}}^{G_E} r_2' = \mathrm{Ind}_{G_{E'}}^{G_E}(\sum_{i=1}^{s} m_i \mathrm{Ind}_{G_{E_i'}}^{G_{E'}} \tau_i) = \sum_{i=1}^{s} m_i \mathrm{Ind}_{G_{E_i'}}^{G_E} \tau_i.$$

Hence, the identities

$$\begin{aligned}
\frac{\mathrm{e}_E(\mathrm{Ind}_{G_{E'}}^{G_E} r_1', \psi \circ \mathrm{Tr}_{E/K})}{\mathrm{e}_E(\mathrm{Ind}_{G_{E'}}^{G_E} r_2', \psi \circ \mathrm{Tr}_{E/K})} &= \frac{\mathrm{e}_E(\sum_{i=1}^{s} n_i \mathrm{Ind}_{G_{E_i'}}^{G_E} \chi_i, \psi \circ \mathrm{Tr}_{E/K})}{\mathrm{e}_E(\sum_{i=1}^{s} m_i \mathrm{Ind}_{G_{E_i'}}^{G_E} \tau_i, \psi \circ \mathrm{Tr}_{E/K})} \\
&= \frac{\prod_{i=1}^{s} \mathrm{e}_E(\mathrm{Ind}_{G_{E_i'}}^{G_E} \chi_i, \psi \circ \mathrm{Tr}_{E/K})^{n_i}}{\prod_{i=1}^{s} \mathrm{e}_E(\mathrm{Ind}_{G_{E_i'}}^{G_E} \tau_i, \psi \circ \mathrm{Tr}_{E/K})^{m_i}} \\
&= \frac{\prod_{i=1}^{s} \mathrm{e}_{E_i'}(\chi_i, \psi \circ \mathrm{Tr}_{E_i'/K})^{n_i}}{\prod_{i=1}^{s} \mathrm{e}_{E_i'}(\tau_i, \psi \circ \mathrm{Tr}_{E_i'/K})^{m_i}}
\end{aligned}$$

and

$$\frac{\mathrm{e}_{E'}(r_1',\psi\circ\mathrm{Tr}_{E'/K})}{\mathrm{e}_{E'}(r_2',\psi\circ\mathrm{Tr}_{E'/K})}=\frac{\mathrm{e}_{E'}(\sum_{i=1}^{s}n_i\mathrm{Ind}_{G_{E_i'}}^{G_{E'}}\chi_i,\psi\circ\mathrm{Tr}_{E'/K})}{\mathrm{e}_{E'}(\sum_{i=1}^{s}m_i\mathrm{Ind}_{G_{E_i'}}^{G_{E'}}\tau_i,\psi\circ\mathrm{Tr}_{E'/K})}$$
$$=\frac{\prod_{i=1}^{s}\mathrm{e}_{E'}(\mathrm{Ind}_{G_{E_i'}}^{G_{E'}}\chi_i,\psi\circ\mathrm{Tr}_{E'/K})^{n_i}}{\prod_{i=1}^{s}\mathrm{e}_{E'}(\mathrm{Ind}_{G_{E_i'}}^{G_{E'}}\tau_i,\psi\circ\mathrm{Tr}_{E'/K})^{m_i}}$$
$$=\frac{\prod_{i=1}^{s}\mathrm{e}_{E_i'}(\chi_i,\psi\circ\mathrm{Tr}_{E_i'/K})^{n_i}}{\prod_{i=1}^{s}\mathrm{e}_{E_i'}(\tau_i,\psi\circ\mathrm{Tr}_{E_i'/K})^{m_i}}$$

follow from Lemma 5.2 at once. Therefore,

$$\frac{\mathrm{e}_{E}(\mathrm{Ind}_{G_{E'}}^{G_E}r_1',\psi\circ\mathrm{Tr}_{E/K})}{\mathrm{e}_{E}(\mathrm{Ind}_{G_{E'}}^{G_E}r_2',\psi\circ\mathrm{Tr}_{E/K})}=\frac{\mathrm{e}_{E'}(r_1',\psi\circ\mathrm{Tr}_{E'/K})}{\mathrm{e}_{E'}(r_2',\psi\circ\mathrm{Tr}_{E'/K})},$$

which proves the following theorem:

Theorem 5.3. *Assume that $K\subseteq E\subseteq E'$ is a tower of finite and separable extensions of K. Fix a non-trivial additive character $\psi:K^+\to\mathbb{C}^\times$ of K. Let*

$$r_1',r_2':G_{E'}\to\mathrm{GL}(V)$$

be two continuous representations of $G_{E'}$ on an n-dimensional space V over $\mathbb{C}$. Then,

$$\frac{\mathrm{e}_{E}(\mathrm{Ind}_{G_{E'}}^{G_E}r_1',\psi\circ\mathrm{Tr}_{E/K})}{\mathrm{e}_{E}(\mathrm{Ind}_{G_{E'}}^{G_E}r_2',\psi\circ\mathrm{Tr}_{E/K})}=\frac{\mathrm{e}_{E'}(r_1',\psi\circ\mathrm{Tr}_{E'/K})}{\mathrm{e}_{E'}(r_2',\psi\circ\mathrm{Tr}_{E'/K})}.$$

Note that, the assignment

$$(V,\psi)\rightsquigarrow\mathrm{e}_K(V,\psi)$$

which associates with each pair (V,ψ), consisting of a continuous representation of G_K on an n-dimensional vector space V over $\mathbb{C}$ and a non-trivial additive character $\psi:K^+\to\mathbb{C}^\times$ of K, a non-zero number $\mathrm{e}_K(V,\psi)\in\mathbb{C}^\times$ has the following properties:

(i) If $\dim(V)=1$ and $\chi:G_K\to\mathbb{C}^\times$ is the corresponding quasi-character, then

$$\mathrm{e}_K(V,\psi)=\varepsilon_K^{\mathrm{Tate}}(\chi,\psi),$$

where $d\mu^{\nabla_K^{\mathrm{ab}}}$ corresponds to the additive Haar measure $d\mu_\psi$ on K^+;

(ii) For any finite separable extension E/K, the mapping

$$(V,\psi\circ\mathrm{Tr}_{E/K})\rightsquigarrow\mathrm{e}_E(V,\psi\circ\mathrm{Tr}_{E/K})$$

which associates with each pair $(V,\psi\circ\mathrm{Tr}_{E/K})$, consisting of a continuous representation of G_E on an n-dimensional vector space V over $\mathbb{C}$ and a non-trivial additive character $\psi:K^+\to\mathbb{C}^\times$ of K, a non-zero number $\mathrm{e}_E(V,\psi\circ\mathrm{Tr}_{E/K})\in\mathbb{C}^\times$ is inductive in degree 0:

(a) *Additivity*– For any short exact sequence

$$0\to V'\to V\to V''\to 0$$

of continuous representations of G_E,

$$\mathrm{e}_E(V,\psi\circ\mathrm{Tr}_{E/K})=\mathrm{e}_E(V',\psi\circ\mathrm{Tr}_{E/K})\mathrm{e}_E(V'',\psi\circ\mathrm{Tr}_{E/K});$$

(b) For any tower $K \subseteq E \subseteq E'$ of finite separable extensions of K,

$$\frac{\mathrm{e}_E(\mathrm{Ind}_{G_{E'}}^{G_E} r_1', \psi \circ \mathrm{Tr}_{E/K})}{\mathrm{e}_E(\mathrm{Ind}_{G_{E'}}^{G_E} r_2', \psi \circ \mathrm{Tr}_{E/K})} = \frac{\mathrm{e}_{E'}(r_1', \psi \circ \mathrm{Tr}_{E'/K})}{\mathrm{e}_{E'}(r_2', \psi \circ \mathrm{Tr}_{E'/K})},$$

where

$$r_1', r_2' : G_{E'} \to \mathrm{GL}(V)$$

are any two continuous representations of $G_{E'}$ on an n-dimensional space V over $\mathbb{C}$.

Thus, combining with Deligne, Dwork, and Langlands' Theorem, next theorem follows immediately:

Theorem 5.4. *Langlands' local ε-factor $\varepsilon_K^{\mathrm{Langlands}}(V,\psi)$ for the pair (V,ψ) consisting of a continuous representation of G_K on an n-dimensional vector space V over $\mathbb{C}$ and a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$\varepsilon_K^{\mathrm{Langlands}}(V,\psi) = \mathrm{e}_K(V,\psi).$$

Recall that, the local root number $W_K(V,\psi)$ of a continuous n-dimensional representation V of G_K over $\mathbb{C}$ with respect to the non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is defined by

$$W_K(V,\psi) = \frac{\varepsilon_K^{\mathrm{Langlands}}(V,\psi)}{|\ \varepsilon_K^{\mathrm{Langlands}}(V,\psi)\ |}.$$

Thus, the following corollary directly follows from Theorem 5.4.

Corollary 5.5. *The local root number $W_K(V,\psi)$ of a continuous n-dimensional representation V of G_K over $\mathbb{C}$ with respect to a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$W_K(V,\psi) = \frac{\mathrm{D}_K(V,\psi,d,c)\mathrm{I}_K(V,\psi,d,c)}{|\ \mathrm{D}_K(V,\psi,d,c)\mathrm{I}_K(V,\psi,d,c)\ |},$$

where $d \in \mathbb{Z}$ satisfying $\partial_K(V) \le d$ and $c \in K^\times$ satisfying $v_K(c) = a(V) + n(\psi)$.

6. An explicit description of ε-factors of Weil-Deligne representations over $\mathbb{C}$

Fix a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K.

First, let $\rho : W_K \to \mathrm{GL}(V)$ be a continuous semisimple representation of W_K on an n-dimensional space V over $\mathbb{C}$. Let

$$V = V_1 \oplus \cdots \oplus V_k$$

be the decomposition of the representation V of W_K into irreducible representations of W_K. Then, by the additivity of $\varepsilon_K^{\mathrm{Langlands}}(V,\psi)$ as a function of V, and by (3.5), the following identities

$$\begin{aligned}\varepsilon_K^{\mathrm{Langlands}}(V,\psi) &= \varepsilon_K^{\mathrm{Langlands}}(V_1,\psi)\cdots\varepsilon_K^{\mathrm{Langlands}}(V_k,\psi)\\ &= \varepsilon_K^{\mathrm{Langlands}}(V_1^{\mathrm{Gal}}\otimes\omega_{s_1},\psi)\cdots\varepsilon_K^{\mathrm{Langlands}}(V_k^{\mathrm{Gal}}\otimes\omega_{s_k},\psi)\\ &= \varepsilon_K^{\mathrm{Langlands}}(V_1^{\mathrm{Gal}},\psi)q_K^{-s_1[a(V_1^{\mathrm{Gal}})+n(\psi)\dim(V_1^{\mathrm{Gal}})]}\cdots\varepsilon_K^{\mathrm{Langlands}}(V_k^{\mathrm{Gal}},\psi)q_K^{-s_k[a(V_k^{\mathrm{Gal}})+n(\psi)\dim(V_k^{\mathrm{Gal}})]}\\ &= \varepsilon_K^{\mathrm{Langlands}}(V_1^{\mathrm{Gal}},\psi)\cdots\varepsilon_K^{\mathrm{Langlands}}(V_k^{\mathrm{Gal}},\psi)q_K^{-\sum_{j=1}^k s_j[a(V_j^{\mathrm{Gal}})+n(\psi)\dim(V_j^{\mathrm{Gal}})]},\end{aligned}$$

follow immediately. Therefore, combining with Theorem 5.4, we have the following theorem:

Theorem 6.1. *Langlands' local ε-factor $\varepsilon_K^{\text{Langlands}}(V,\psi)$ for the pair (V,ψ) consisting of a continuous semisimple representation of W_K on an n-dimensional vector space V over $\mathbb{C}$ and a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$\varepsilon_K^{\text{Langlands}}(V,\psi) = \mathrm{e}_K(V_1^{\text{Gal}},\psi)\cdots \mathrm{e}_K(V_k^{\text{Gal}},\psi) q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]},$$

where $V = V_1 \oplus \cdots \oplus V_k$ is the decomposition of the representation V of W_K into irreducible representations V_j of W_K, with $V_j = V_j^{\text{Gal}} \otimes \omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j \in \mathbb{C}$, for $j = 1,\cdots,k$.

Recall that, the local root number $W_K(V,\psi)$ of a continuous semisimple n-dimensional representation V of W_K over $\mathbb{C}$ with respect to the non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is defined by

$$W_K(V,\psi) = \frac{\varepsilon_K^{\text{Langlands}}(V,\psi)}{\mid \varepsilon_K^{\text{Langlands}}(V,\psi) \mid}.$$

Thus, the following corollary directly follows from Theorem 6.1.

Corollary 6.2. *The local root number $W_K(V,\psi)$ of a continuous semisimple n-dimensional representation V of W_K over $\mathbb{C}$ with respect to a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$W_K(V,\psi) = W_K(V_1^{\text{Gal}},\psi)\cdots W_K(V_k^{\text{Gal}},\psi) \frac{q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}}{\mid q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]} \mid}$$

where $V = V_1 \oplus \cdots \oplus V_k$ is the decomposition of the representation V of W_K into irreducible representations V_j of W_K, with $V_j = V_j^{\text{Gal}} \otimes \omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j \in \mathbb{C}$, for $j = 1,\cdots,k$.

Now, let (V,N) be a Weil-Deligne representation of WD_K over $\mathbb{C}$. Without loss of generality, we can assume that (V,N) is a Φ-semisimple representation of WD_K over $\mathbb{C}$ as

$$\varepsilon_K^{\text{Langlands}}((V,N),\psi) = \varepsilon_K^{\text{Langlands}}((V,N)_{\text{ss}},\psi).$$

Thus, assume that (V,N) is a Φ-semisimple representation of WD_K over $\mathbb{C}$. Then, as V is a semisimple representation of W_K, combining Theorem 6.1 with the definition of the Langlands' local ε-factor $\varepsilon_K^{\text{Langlands}}((V,N),\psi)$ of the Weil-Deligne representation (V,N) with respect to ψ:

$$\varepsilon_K^{\text{Langlands}}((V,N),\psi) = \varepsilon_K^{\text{Langlands}}(V,\psi).\det(-\Phi \mid_{V^{I_K}/V_N^{I_K}}),$$

we obtain the following theorem on explicit description of Langlands' ε-factors of Weil-Deligne representations of WD_K over $\mathbb{C}$ with respect to ψ:

Theorem 6.3. *Langlands' local ε-factor $\varepsilon_K^{\text{Langlands}}((V,N),\psi)$ for the pair $((V,N),\psi)$ consisting of a Φ-semisimple Weil-Deligne representation (V,N) of WD_K over $\mathbb{C}$ and a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$\varepsilon_K^{\text{Langlands}}((V,N),\psi) = \mathrm{e}_K(V_1^{\text{Gal}},\psi)\cdots \mathrm{e}_K(V_k^{\text{Gal}},\psi) q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}.\det(-\Phi \mid_{V^{I_K}/V_N^{I_K}}),$$

where $V = V_1 \oplus \cdots \oplus V_k$ is the decomposition of the representation V of W_K into irreducible representations V_j of W_K, with $V_j = V_j^{\text{Gal}} \otimes \omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j \in \mathbb{C}$, for $j = 1,\cdots,k$.

Recall that, every indecomposable Weil-Deligne representation (V,N) of WD_K over $\mathbb{C}$ has the form

$$(V,N) = (V_o,N_o)\otimes \mathrm{Sp}(d),$$

where (V_o,N_o) is an irreducible representation of WD_K over $\mathbb{C}$, which means that V_o is an irreducible representation of W_K over $\mathbb{C}$ and $N_o = 0$, and $\mathrm{Sp}(d)$ is the special representation of WD_K over $\mathbb{C}$ for some $1 \le d \in \mathbb{Z}$. Then the underlying Weil group representation of $(V_o,N_o)\otimes \mathrm{Sp}(d)$ over $\mathbb{C}$ is $V_o\oplus V_o(1)\oplus\cdots\oplus V_o(d-1)$, where $V_o(j)$ denotes $V_o\otimes\omega_j$ for $j = 1,\cdots,d-1$.

Theorem 6.4. *Let (V,N) be an indecomposable representation of WD_K over $\mathbb{C}$ with the decomposition*

$$(V,N) = (V_o,N_o)\otimes \mathrm{Sp}(d),$$

where (V_o,N_o) is an irreducible representation of WD_K over $\mathbb{C}$ and $\mathrm{Sp}(d)$ is the special representation of WD_K over $\mathbb{C}$ for some $1 \le d \in \mathbb{Z}$. Then, Langlands' local ε-factor $\varepsilon_K^{\mathrm{Langlands}}((V,N),\psi)$ for the pair $((V,N),\psi)$, where $\psi : K^+ \to \mathbb{C}^\times$ denotes a non-trivial additive character of K is given by

$$\varepsilon_K^{\mathrm{Langlands}}((V,N),\psi) = \left(\mathrm{e}_K(V_o^{\mathrm{Gal}},\psi)q_K^{-s_o[a(V_o^{\mathrm{Gal}})+n(\psi)\dim(V_o^{\mathrm{Gal}})]}\right)^d .q_K^{-\frac{(d-1)d}{2}(n(\psi)\dim(V_o)+a(V_o))}.\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}}),$$

where $V_o = V_o^{\mathrm{Gal}}\otimes\omega_{s_o}$ for some Galois type representation V_o^{Gal} of W_K and for some $s_o\in\mathbb{C}$.

Proof. Let (V,N) be an indecomposable representation of WD_K over $\mathbb{C}$ with the decomposition

$$(V,N) = (V_o,N_o)\otimes \mathrm{Sp}(d),$$

where (V_o,N_o) is an irreducible representation of WD_K over $\mathbb{C}$ and $\mathrm{Sp}(d)$ is the special representation of WD_K over $\mathbb{C}$ for some $1 \le d \in \mathbb{Z}$. The underlying Weil group representation of $(V_o,N_o)\otimes \mathrm{Sp}(d)$ over $\mathbb{C}$ is $V_o\oplus V_o(1)\oplus\cdots\oplus V_o(d-1)$, where $V_o(j)$ denotes $V_o\otimes\omega_j$ for $j = 1,\cdots,d-1$. Therefore,

$$\begin{aligned}\varepsilon_K^{\mathrm{Langlands}}((V,N),\psi) &= \varepsilon_K^{\mathrm{Langlands}}((V_o,N_o)\otimes\mathrm{Sp}(d),\psi)\\ &= \varepsilon_K^{\mathrm{Langlands}}(V_o,\psi)\varepsilon_K^{\mathrm{Langlands}}(V_o(1),\psi)\cdots\varepsilon_K^{\mathrm{Langlands}}(V_o(d-1),\psi).\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}})\\ &= \varepsilon_K^{\mathrm{Langlands}}(V_o,\psi)\varepsilon_K^{\mathrm{Langlands}}(V_o\otimes\omega_1,\psi)\cdots\varepsilon_K^{\mathrm{Langlands}}(V_o\otimes\omega_{d-1},\psi).\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}})\\ &= \varepsilon_K^{\mathrm{Langlands}}(V_o,\psi)^d.q_K^{-\frac{(d-1)d}{2}(n(\psi)\dim(V_o)+a(V_o))}.\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}}),\end{aligned}$$

by part (ii) in Section 2.2. Now, by Theorem 6.1,

$$\begin{aligned}\varepsilon_K^{\mathrm{Langlands}}((V,N),\psi) &= \varepsilon_K^{\mathrm{Langlands}}(V_o,\psi)^d.q_K^{-\frac{(d-1)d}{2}(n(\psi)\dim(V_o)+a(V_o))}.\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}})\\ &= \left(\mathrm{e}_K(V_o^{\mathrm{Gal}},\psi)q_K^{-s_o[a(V_o^{\mathrm{Gal}})+n(\psi)\dim(V_o^{\mathrm{Gal}})]}\right)^d .q_K^{-\frac{(d-1)d}{2}(n(\psi)\dim(V_o)+a(V_o))}.\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}}),\end{aligned}$$

where $V_o = V_o^{\mathrm{Gal}}\otimes\omega_{s_o}$ for some Galois type representation V_o^{Gal} of W_K and for some $s_o \in \mathbb{C}$. □

Given a Weil-Deligne representation (V,N) of WD_K over $\mathbb{C}$. Following [14], define the local root number $W_K((V,N),\psi)$ of (V,N) with respect to the additive character $\psi : K^\times \to \mathbb{C}^\times$ by

$$W_K((V,N),\psi) = \frac{\varepsilon_K^{\mathrm{Langlands}}((V,N),\psi)}{\mid \varepsilon_K^{\mathrm{Langlands}}((V,N),\psi)\mid}.$$

Then, we have the following corollary of Theorem 6.3.

Corollary 6.5. *The local root number $W_K((V,N),\psi)$ of a Φ-semisimple Weil-Deligne representation (V,N) of WD_K over $\mathbb{C}$ with respect to a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$W_K((V,N),\psi) = W_K(V_1^{\mathrm{Gal}},\psi)\cdots W_K(V_k^{\mathrm{Gal}},\psi)\frac{q_K^{-\sum_{j=1}^k s_j[a(V_j^{\mathrm{Gal}})+n(\psi)\dim(V_j^{\mathrm{Gal}})]}.\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}})}{\mid q_K^{-\sum_{j=1}^k s_j[a(V_j^{\mathrm{Gal}})+n(\psi)\dim(V_j^{\mathrm{Gal}})]}.\det(-\Phi\mid_{V^{I_K}/V_N^{I_K}})\mid},$$

where $V = V_1 \oplus \cdots \oplus V_k$ is the decomposition of the representation V of W_K into irreducible representations V_j of W_K, with $V_j = V_j^{\mathrm{Gal}} \otimes \omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j \in \mathbb{C}$, for $j = 1,\cdots,k$.

7. A FORMULA FOR ε-FACTORS OF ADMISSIBLE REPRESENTATIONS OF WA_K OVER $\mathbb{C}$

There is an obvious analogue of Theorem 6.3 for admissible representations of the Weil-Arthur group WA_K of K over $\mathbb{C}$.

Theorem 7.1. *Langlands' local ε-factor $\varepsilon_K^{\mathrm{Langlands}}(\varphi,\psi)$ for the pair (φ,ψ) consisting of an admissible representation $\varphi : WA_K \to \mathrm{GL}(V)$ of WA_K on an n-dimensional vector space V over $\mathbb{C}$ and a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$\varepsilon_K^{\mathrm{Langlands}}(\varphi,\psi) = \mathrm{e}_K(V_1^{\mathrm{Gal}},\psi)\cdots\mathrm{e}_K(V_k^{\mathrm{Gal}},\psi)q_K^{-\sum_{j=1}^k s_j[a(V_j^{\mathrm{Gal}})+n(\psi)\dim(V_j^{\mathrm{Gal}})]}.\det(-\Phi\mid_{(V_{\rho_\varphi})^{I_K}/(V_{\rho_\varphi})_{N_\varphi}^{I_K}}),$$

where (ρ_φ,N_φ) is the Φ-semisimple Weil-Deligne representation of WD_K over $\mathbb{C}$ corresponding to φ, and $V_{\rho_\varphi} = V_1 \oplus \cdots \oplus V_k$ is the decomposition of the representation V_{ρ_φ} of W_K into irreducible representations V_j of W_K, with $V_j = V_j^{\mathrm{Gal}} \otimes \omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j \in \mathbb{C}$, for $j = 1,\cdots,k$.

Let $\varphi : WA_K \to \mathrm{GL}(V)$ be an admissible representation of WA_K on an n-dimensional vector space V over $\mathbb{C}$. Assume that

$$V_\varphi = V_o \otimes V^{(d)},$$

where V_o is an irreducible representation of W_K over $\mathbb{C}$ and $V^{(d)}$ is the unique d-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ over $\mathbb{C}$. Then, the corresponding Φ-semisimple Weil-Deligne representation (ρ_φ,N_φ) of WD_K over $\mathbb{C}$ is indecomposable and has the form

$$(\rho_\varphi,N_\varphi) = (V_o,0) \otimes \mathrm{Sp}(d),$$

where $(V_o,0)$ is an irreducible representation of WD_K over $\mathbb{C}$ and $\mathrm{Sp}(d)$ is the special representation of WD_K over $\mathbb{C}$ for this particular $1 \le d \in \mathbb{Z}$. Then the underlying Weil group representation V_{ρ_φ} of $(V_o,0)\otimes\mathrm{Sp}(d)$ over $\mathbb{C}$ is $V_o \oplus V_o(1) \oplus \cdots \oplus V_o(d-1)$, where $V_o(j)$ denotes $V_o \otimes \omega_j$ for $j = 1,\cdots,d-1$. The analogue of Theorem 6.4 for admissible representations of the Weil-Arthur group WA_K of K is the following result:

Theorem 7.2. *Let $\varphi : WA_K \to \mathrm{Gl}(V)$ be an admissible representation of WA_K on an n-dimensional space V over $\mathbb{C}$ with the decomposition*

$$V_\varphi = V_o \otimes V^{(d)}$$

where V_o is an irreducible representation of W_K over $\mathbb{C}$ and $V^{(d)}$ is the unique d-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ over $\mathbb{C}$ for some $1 \le d \in \mathbb{Z}$. Then, Langlands' local

ε-factor $\varepsilon_K^{\text{Langlands}}(\varphi,\psi)$ for the pair (φ,ψ), where $\psi : K^+ \to \mathbb{C}^\times$ denotes a non-trivial additive character of K is given by

$$\varepsilon_K^{\text{Langlands}}(\varphi,\psi) = \left(\mathrm{e}_K(V_o^{\text{Gal}},\psi)q_K^{-s_o[a(V_o^{\text{Gal}})+n(\psi)\dim(V_o^{\text{Gal}})]}\right)^d .q_K^{-\frac{(d-1)d}{2}(n(\psi)\dim(V_o)+a(V_o))}.\det(-\Phi\mid_{(V_{\rho_\varphi})^{I_K}/(V_{\rho_\varphi})^{I_K}_{N_\varphi}}),$$

where $V_o = V_o^{\text{Gal}} \otimes \omega_{s_o}$ for some Galois type representation V_o^{Gal} of W_K and for some $s_o \in \mathbb{C}$, and $(\rho_\varphi, N_\varphi)$ is the Φ-semisimple Weil-Deligne representation of WD_K over $\mathbb{C}$ corresponding to φ.

Given an admissible representation $\varphi : WA_K \to \mathrm{GL}(V)$ of WA_K on an n-dimensional vector space V over $\mathbb{C}$. By the definition of $\varepsilon_K^{\text{Langlands}}(\varphi,\psi)$ given in (4.1), define the local root number $W_K(\varphi,\psi)$ of φ with respect to the additive character $\psi : K^\times \to \mathbb{C}^\times$ by

$$W_K(\varphi,\psi) = W_K((\rho_\varphi,N_\varphi),\psi).$$

Then, we have the following corollary of Theorem 7.1.

Corollary 7.3. *The local root number $W_K(\varphi,\psi)$ of an admissible representation $\varphi : WA_K \to \mathrm{GL}(V)$ of WA_K on an n-dimensional vector space V over $\mathbb{C}$ with respect to a non-trivial additive character $\psi : K^+ \to \mathbb{C}^\times$ of K is given by*

$$W_K(\varphi,\psi) = W_K(V_1^{\text{Gal}},\psi)\cdots W_K(V_k^{\text{Gal}},\psi)\frac{q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}.\det(-\Phi\mid_{(V_{\rho_\varphi})^{I_K}/(V_{\rho_\varphi})^{I_K}_{N_\varphi}})}{\mid q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}.\det(-\Phi\mid_{(V_{\rho_\varphi})^{I_K}/(V_{\rho_\varphi})^{I_K}_{N_\varphi}})\mid},$$

where $(\rho_\varphi, N_\varphi)$ is the Φ-semisimple Weil-Deligne representation of WD_K over $\mathbb{C}$ corresponding to φ, and $V_{\rho_\varphi} = V_1 \oplus \cdots \oplus V_k$ is the decomposition of the representation V_{ρ_φ} of W_K into irreducible representations V_j of W_K, with $V_j = V_j^{\text{Gal}} \otimes \omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j \in \mathbb{C}$, for $j = 1,\cdots,k$.

8. ε-FACTORS OF LANGLANDS PARAMETERS

The main reference for this part is [3].

8.1. **ε-factors of Weil-Deligne homomorphisms of WD_K in $\mathscr{G}$.** Following [3], a triple $(\rho,\mathscr{G},N)$ consisting of a complex Lie group $\mathscr{G}$ with reductive identity component $\mathscr{G}^o$, a homomorphism

$$\rho : W_K \to \mathscr{G},$$

which is continuous on the inertia subgroup I_K of W_K and $\rho(\Phi_K)$ semisimple in $\mathscr{G}$, and N a nilpotent element in the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$, such that

$$\mathrm{Ad}\rho(w)N = \| w \|_K N,$$

for every $w \in W_K$ is called a *Weil-Deligne homomorphism of WD_K in $\mathscr{G}$*. Two Weil-Deligne homomorphisms $(\rho,\mathscr{G},N)$ and $(\rho',\mathscr{G},N')$ of WD_K in $\mathscr{G}$ are said to be equivalent, if there exists $g \in \mathscr{G}^o$ such that $\rho' = g\rho g^{-1}$ and $N' = \mathrm{Ad}(g)N$.

Fix an additive character $\psi : K^+ \to \mathbb{C}^\times$ of K. For any finite-dimensional representation

$$r : \mathscr{G} \to \mathrm{GL}(V)$$

of the Lie group $\mathscr{G}$ on an n-dimensional vector space V over $\mathbb{C}$, which is furthermore assumed to be algebraic on $\mathscr{G}^o$, define Langlands' local constant $\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi)$ of the Weil-Deligne homomorphism $(\rho,\mathscr{G},N)$ of WD_K in $\mathscr{G}$ with respect to the representation r of $\mathscr{G}$ on V over $\mathbb{C}$ and with respect to ψ by

$$\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi)=\varepsilon_K^{\text{Langlands}}\left((V_{r\circ\rho},N),\psi\right),$$

where $(V_{r\circ\rho},dr(N))$ is an n-dimensional Φ-semisimple Weil-Deligne representation of WD_K on V over $\mathbb{C}$, because $r\circ\rho(\Phi_K)$ is semisimple in $\mathrm{GL}(V)$, and

$$\mathrm{Ad}(r\circ\rho)(w)dr(N)=||\, w\,||_K\, dr(N),$$

for every $w\in W_K$. Thus, by equations (3.1), (3.2), and (3.3), define

$$\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi)=\varepsilon_K^{\text{Langlands}}\left(V_{r\circ\rho},\psi\right).\det(-\Phi\mid_{V_{r\circ\rho}^{I_K}/(V_{r\circ\rho})_{dr(N)}^{I_K}}),\tag{8.1}$$

$$\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi;t)=\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi)\,t^{a((V_{r\circ\rho},dr(N)))},\tag{8.2}$$

and

$$\varepsilon_K^{\text{Langlands}}(s,(\rho,\mathscr{G},N),r,\psi)=\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi;t)\mid_{t=q^{-s}}\tag{8.3}$$

respectively.

Reformulating Theorem 6.3, we get the following theorem:

Theorem 8.1. *Langlands' local ε-factor $\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi)$ of the Weil-Deligne homomorphism of WD_K in $\mathscr{G}$ with respect to the representation r of $\mathscr{G}$ on V over $\mathbb{C}$ and with respect to ψ is given by*

$$\varepsilon_K^{\text{Langlands}}((\rho,\mathscr{G},N),r,\psi)=\mathrm{e}_K(V_1^{\text{Gal}},\psi)\cdots\mathrm{e}_K(V_k^{\text{Gal}},\psi)q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}.\det(-\Phi\mid_{V_{r\circ\rho}^{I_K}/(V_{r\circ\rho})_{dr(N)}^{I_K}}),$$

where $V_{r\circ\rho}=V_1\oplus\cdots\oplus V_k$ is the decomposition of the representation $V_{r\circ\rho}$ of W_K into irreducible representations V_j of W_K, with $V_j=V_j^{\text{Gal}}\otimes\omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j\in\mathbb{C}$, for $j=1,\cdots,k$.

8.2. **Admissible homomorphisms of WA_K into $\mathscr{G}$ and their ε-factors.** Let $\mathscr{G}$ be a complex Lie group with reductive identity component $\mathscr{G}^o$. A homomorphism

$$\varphi: WA_K\to\mathscr{G}$$

is called an admissible homomorphism of the Weil-Arthur group WA_K into $\mathscr{G}$, if (i) φ is trivial on an open subgroup of the inertia group I_K of K; (ii) $\varphi(\Phi_K)$ is semisimple; and (iii) $\varphi\mid_{\mathrm{SL}(2,\mathbb{C})}$ is an algebraic homomorphism. Two admissible homomorphisms φ_1 and φ_2 of WA_K in $\mathscr{G}$ are equivalent if φ_1 and φ_2 are $\mathscr{G}^o$-conjugate.

Following the same lines of reasoning of Section 4.1 (look at 2.1 of [3] for details), there exists a bijective correspondence

$$(\rho,\mathscr{G},N)\rightsquigarrow\varphi$$

between the equivalence classes of Weil-Deligne homomorphisms $(\rho,\mathscr{G},N)$ of WD_K in a complex Lie group $\mathscr{G}$ and the $\mathscr{G}^o$-conjugacy classes of admissible homomorphisms φ of the Weil-Arthur group WA_K of K in $\mathscr{G}$.

Fix an additive character $\psi: K^+\to\mathbb{C}^\times$ of K. For any finite-dimensional representation

$$r:\mathscr{G}\to\mathrm{GL}(V)$$

of the Lie group $\mathscr{G}$ on an n-dimensional vector space V over $\mathbb{C}$, which is furthermore assumed to be algebraic on $\mathscr{G}^o$, define Langlands' local constant $\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)$ of the admissible homomorphism φ of WA_K in $\mathscr{G}$ with respect to the representation r of $\mathscr{G}$ on V over $\mathbb{C}$ and with respect to ψ by

$$\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)=\varepsilon_K^{\text{Langlands}}\left((\rho_\varphi,\mathscr{G},N_\varphi),r,\psi\right), \tag{8.4}$$

where $(\rho_\varphi,\mathscr{G},N_\varphi)$ is the Weil-Deligne homomorphism of WD_K in $\mathscr{G}$ corresponding to the admissible homomorphism φ of WA_K in $\mathscr{G}$, and further set

$$\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi;t)=\varepsilon_K^{\text{Langlands}}\left((\rho_\varphi,\mathscr{G},N_\varphi),r,\psi;t\right), \tag{8.5}$$

and

$$\varepsilon_K^{\text{Langlands}}(s,\varphi,r,\psi)=\varepsilon_K^{\text{Langlands}}\left(s,(\rho_\varphi,\mathscr{G},N_\varphi),r,\psi\right) \tag{8.6}$$

respectively.

Reformulating Theorem 8.1, we get the following theorem:

Theorem 8.2. *Langlands' local ε-factor $\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)$ of the admissible homomorphism φ of WA_K in $\mathscr{G}$ with respect to the representation r of $\mathscr{G}$ on V over $\mathbb{C}$ and with respect to ψ is given by*

$$\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)=\mathrm{e}_K(V_1^{\text{Gal}},\psi)\cdots\mathrm{e}_K(V_k^{\text{Gal}},\psi)q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}.\det(-\Phi\mid_{V_{r\circ\rho_\varphi}^{I_K}/(V_{r\circ\rho_\varphi})_{dr(N_\varphi)}^{I_K}}),$$

where $V_{r\circ\rho_\varphi}=V_1\oplus\cdots\oplus V_k$ is the decomposition of the representation $V_{r\circ\rho_\varphi}$ of W_K into irreducible representations V_j of W_K, with $V_j=V_j^{\text{Gal}}\otimes\omega_{s_j}$ for some Galois type representation V_j^{Gal} of W_K and $s_j\in\mathbb{C}$, for $j=1,\cdots,k$.

8.3. **ε-factors of local Langlands parameters of** G. The following setting is central in Langlands reciprocity and functoriality principles. Assume that G is a connected reductive algebraic group over the local field K. The L-group ${}^L\mathrm{G}$ of G is a complex Lie group with a reductive connected component ${}^L\mathrm{G}^o=\widehat{\mathrm{G}}$ over $\mathbb{C}$.

Recall that, an admissible homomorphism

$$\varphi: WA_K\to {}^L\mathrm{G}$$

of the Weil-Arthur group WA_K in the L-group ${}^L\mathrm{G}$ of G is called a local Langlands parameter of G. From Section 8.2, there exists a unique Weil-Deligne homorphism $(\rho_\varphi,{}^L\mathrm{G},N_\varphi)$ of WD_K in ${}^L\mathrm{G}$ corresponding to the local parameter φ.

Now, fix an additive character $\psi:K^+\to\mathbb{C}^\times$ of K. For any finite-dimensional representation

$$r:{}^L\mathrm{G}\to\mathrm{GL}(V)$$

of the L-group ${}^L\mathrm{G}$ of G on an n-dimensional vector space V over $\mathbb{C}$, which is furthermore assumed to be algebraic on ${}^L\mathrm{G}^o$, Langlands' local constant $\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)$ of a local Langlands parameter $\varphi:WA_K\to{}^L\mathrm{G}$ of $\mathscr{G}$ with respect to the representation r of ${}^L\mathrm{G}$ on V over $\mathbb{C}$ and with respect to ψ has an explicit expression, which is the content of the following corollary of Theorem 8.2:

Corollary 8.3. *Langlands' local ε-factor $\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)$ of the local Langlands parameter $\varphi:WA_K\to{}^L\mathrm{G}$ with respect to the representation r of ${}^L\mathrm{G}$ on V over $\mathbb{C}$ and with respect to ψ is given by*

$$\varepsilon_K^{\text{Langlands}}(\varphi,r,\psi)=\mathrm{e}_K(V_1^{\text{Gal}},\psi)\cdots\mathrm{e}_K(V_k^{\text{Gal}},\psi)q_K^{-\sum_{j=1}^k s_j[a(V_j^{\text{Gal}})+n(\psi)\dim(V_j^{\text{Gal}})]}.\det(-\Phi\mid_{V_{r\circ\rho_\varphi}^{I_K}/(V_{r\circ\rho_\varphi})_{dr(N_\varphi)}^{I_K}}),$$

where $V_{r\circ\rho_\varphi} = V_1 \oplus \cdots \oplus V_k$ *is the decomposition of the representation* $V_{r\circ\rho_\varphi}$ *of* W_K *into irreducible representations* V_j *of* W_K*, with* $V_j = V_j^{\mathrm{Gal}} \otimes \omega_{s_j}$ *for some Galois type representation* V_j^{Gal} *of* W_K *and* $s_j \in \mathbb{C}$*, for* $j = 1, \cdots, k$*.*

REFERENCES

[1] P. Deligne, *Les contantes des équations fonctionelles des fonctions L*, Modular functions of one variable II (Proc. Internat. Summer School, Univ. Antwerp, 17 July-3 August 1972), P. Deligne and W. Kuyk (eds.), Lecture Notes in Mathematics, **349**, Springer-Verlag, Berlin-Heidelberg-New York, 1973, 501–597. 2, 3

[2] P. X. Gallagher, *Determinants of representations of finite groups*, Abh. Math. Sem. Univ. Hamburg, **28** (3–4), 1965, 162–167.

[3] B. Gross and M. Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke. Math. Journal, **154** (3), 2010, 431–508. 22, 23

[4] K. I. Ikeda, *On the non-abelian global class field theory*, Annales Math. Quebec, **37** (2), 2013, 129–172. 3, 11

[5] K. I. Ikeda, *On the local Artin conductor* $\mathfrak{f}_{\mathrm{Artin}}(\chi)$ *of a character* χ *of* $\mathrm{Gal}(E/K)$ *II: Main results for the metabelian case*, Proc. Indian Acad. Sci. Math. Sci., **113** (2), 2003, 99–137. 7

[6] K. I. Ikeda, S. Kazancıoğlu, and E. Serbest, *On the relationship between the generalized Fesenko and the Laubie reciprocity maps*, submitted. 3, 11

[7] K. I. Ikeda and E. Serbest, *Non-abelian local reciprocity law*, Manuscripta Math., **132**, 2010, 19–49. 3, 10, 12

[8] K. I. Ikeda and E. Serbest, *Ramification theory in non-abelian local class field theory*, Acta Arithmetica, **144**, 2010, 373–393. 3, 10, 11

[9] T. Kaletha, *The local Langlands conjectures for non-quasi-split groups*, preprint.

[10] H. Koch and E. de Shalit, *Metabelian local class field theory*, J. Reine Angew. Math., **478**, 1996, 85–106. 12

[11] R. P. Langlands, *On Artin's L-function*, Rice University Studies, **56** (2), 1970, 23–28. 2, 3

[12] F. Laubie, *Une théorie non abélienne du corps de classes local*, Compositio Math., **143**, 2007, 339–362. 3, 10

[13] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields* (2nd ed.), Grundlehren der mathematischen Wissenschaften, **323**, Springer-Verlag, Berlin and Heidelberg, 2008.

[14] D. E. Rohrlich, *Inductivity of the global root number*, Acta Arithmetica, **159** (3), 2013, 245–256. 20

[15] J. Tate, *Number theoretic background*, Automorphic Forms, Representations, and *L*-functions (A. Borel and W. Casselman eds.), Proc. Sympos. Pure Math., **33** Part 2, Amer. Math. Soc., Providence , R.I., 1979, 3–26. 2, 3, 4, 5, 6

YEDITEPE UNIVERSITY, DEPARTMENT OF MATHEMATICS, 26 AĞUSTOS YERLEŞIMI, İNÖNÜ MAH., KAYIŞDAĞI CAD., 34755, ATAŞEHIR, ISTANBUL, TURKEY

E-mail address: ilhan.ikeda@yeditepe.edu.tr