Basic properties of the non-abelian global reciprocity law

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Abstract. Let K be a global field. The aim of this paper is to study the basic properties of the global non-abelian norm-residue symbol $\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}}$ of K, which is defined following Chevalley-Miyake philosophy of idèles by "glueing" the local non-abelian norm-residue symbol $\{\bullet, K_v\}_{\varphi_v}$ of K_v in the sense of Koch, for each $v \in \mathbb{h}_K$.

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1. Introduction

All through this work K denotes a global field. That is, K is either a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_q(T)$ (i.e., the field of rational functions of a curve defined over a finite field $\mathbb{F}_q$). Let $\mathbb{a}_K$ denote the set of all archimedean primes of K (so in case K is a function field, then $\mathbb{a}_K = \varnothing$) and let $\mathbb{h}_K$ denote the set of all

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henselian (i.e., non-archimedean) primes of K [1] [2]. For each $v \in \mathbb{h}_K \sqcup \mathbb{a}_K$, let K_v denote the completion of K with respect to the v-adic absolute value defined on K.

For any field M, let G_M denote the absolute Galois group $\mathrm{Gal}(M^{sep}/M)$ of M. For any extension A of M, let $(A/M)^{ab}$ denote the maximal abelian extension of M inside A. If G is a topological group, then G^{ab} denotes the abelianization [3] of G. In particular, G_M^{ab} is defined and $G_M^{ab} = \mathrm{Gal}(M^{ab}/M)$, where $M^{ab} = (M^{sep}/M)^{ab}$. In case M is a local or a global field, then W_M denotes the absolute Weil group of M, which comes equipped with a continuous homomorphism $\beta_M : W_M \to G_M$ with dense image. In particular, W_M^{ab} is defined, and there exists a continuous homomorphism $\beta_M^{ab} : W_M^{ab} \to G_M^{ab}$ induced naturally from the continuous arrow $\beta_M : W_M \to G_M$.

1.1. An overview of abelian global class field theory

For the basic theory of local and global fields and class field theory, we refer the reader to [10]. The abelian global class field theory for K establishes a continuous surjective homomorphism

$$(\bullet, K) : \mathbb{J}_K \to W_K^{ab}$$

from the idèle group $\mathbb{J}_K$ of K to the abelianization W_K^{ab} of the absolute Weil group W_K of K, which is called the abelian global reciprocity law of K, or the global abelian norm-residue symbol of K. This arrow satisfies certain "functoriality" and "naturality" conditions, which we shall recall now.

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[1] There is also another disjoint decomposition of the set of all primes $\mathbb{h}_K \sqcup \mathbb{a}_K$ of K, denoted by

$$\mathbb{h}_K \sqcup \mathbb{a}_K = \mathbb{f}_K \sqcup \infty_K,$$

where ∞_K is called the set of primes at infinity of K and $\mathbb{f}_K$ the set of finite primes of K. In case K is a number field, then $\infty_K := \mathbb{a}_K$ and $\mathbb{f}_K := \mathbb{h}_K$ as usual. In the remaining case, namely when K is a global function field, then all primes of K are henselian. However, by Ostrowski's theorem on the classification of all (equivalence classes of) valuations on the global function field $\mathbb{F}_q(T)$, denoting the set of all equivalence classes of valuations on $\mathbb{F}_q(T)$ by $\mathscr{V}_{\mathbb{F}_q(T)}$, there exists a bijective correspondence

$$\mathscr{V}_{\mathbb{F}_q(T)} \rightleftarrows \{p(T) \in \mathbb{F}_q[T] : \ p(T) \text{ monic irred.}_{/\mathbb{F}_q}\} \cup \{\infty\},$$

where ∞ corresponds to the valuation v_∞ on $\mathbb{F}_q(T)$ defined by

$$v_\infty\left(\frac{a(T)}{b(T)}\right) = \deg(a(T)) - \deg(b(T)),$$

for any $0 \neq a(T), b(T) \in \mathbb{F}_q[T]$. Therefore, keeping the analogy between $\mathbb{Q}$ and $\mathbb{F}_q(T)$, we define the set ∞_K consisting of all the primes of K at infinity to be the finite set of all the primes $\infty_1, \cdots, \infty_r$ of K lying over the prime ∞ of $\mathbb{F}_q(T)$. That is, $\infty_1, \cdots, \infty_r$ correspond to the valuations $v_{\infty_1}, \cdots, v_{\infty_r}$ of K lying over the valuation v_∞ of $\mathbb{F}_q(T)$. Here, the inequality

$$\sum_{i=1}^{r} e(\infty_i/\infty) f(\infty_i/\infty) \leq [K : \mathbb{F}_q(T)]$$

is satisfied. So there are at most $[K : \mathbb{F}_q(T)]$ distinct places of K lying over ∞, so $r \leq [K : \mathbb{F}_q(T)]$.

[2] In fact, ∞_K can more generally be defined to be any finite or infinite set consisting of an arbitrary choice $\infty_1, \infty_2, \cdots$ of places (i.e., valuations) of K.

[3] For any topological group G the closure of the 1^{st} commutator subgroup of G is denoted by G^c and the factor group G/G^c, denoted by G^{ab}, is the maximal abelian Hausdorff quotient of G,called the (1^{st}) abelianization of G. The canonical mapping $\mathrm{red}_{G^c} : G \to G/G^c = G^{ab}$ which is defined by reduction modulo G^c is called the abelianization map (or the 1^{st} abelianization map) of G.

First of all, the local and global reciprocity laws $(\bullet, K_v)$ and $(\bullet, K)$ are "compatible" for every $v \in \mathbb{h}_K \cup \mathbb{a}_K$. That is, the following square

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\varepsilon_v} & \mathbb{J}_K \\ {\scriptstyle (\bullet, K_v)}\downarrow & & \downarrow{\scriptstyle (\bullet, K)} \\ W_{K_v}^{ab} & \xrightarrow[e_v^{\mathrm{Weil}^{ab}}]{} & W_K^{ab} \end{array} \tag{1.1}$$

is commutative for each $v \in \mathbb{h}_K \cup \mathbb{a}_K$. Next, if $K \subseteq L \subset E$ is any tower of extensions of global fields, then the following inclusions

$$(L/K)^{ab} \subseteq (E/K)^{ab} \text{ and } (E/K)^{ab} \subseteq (E/L)^{ab}$$

hold. Moreover :

(i) The abelian global reciprocity law $(\bullet, K)$ of K induces a continuous surjective homomorphism

$$(\bullet, L/K) : \mathbb{J}_K \xrightarrow{(\bullet, K)} W_K^{ab} \xrightarrow{\beta_K^{ab}} G_K^{ab} \to \mathrm{Gal}((L/K)^{ab}/K),$$

where $(L/K)^{ab}$ is the maximal abelian extension of K inside L. The kernel of the surjective homomorphism $(\bullet, L/K)$ is $K^\times \mathrm{N}_{L/K} \mathbb{J}_L$. That is, the following sequence

$$1 \to K^\times \mathrm{N}_{L/K} \mathbb{J}_L \to \mathbb{J}_K \xrightarrow{(\bullet, L/K)} \mathrm{Gal}((L/K)^{ab}/K) \to 1$$

is exact.

(ii) For every prime v, the image of the continuous homomorphism defined by the composition

$$K_v^\times \hookrightarrow \mathbb{J}_K \xrightarrow{(\bullet, L/K)} \mathrm{Gal}((L/K)^{ab}/K)$$

is the decomposition group $\mathrm{D}_v((L/K)^{ab}/K)$ of v in $\mathrm{Gal}((L/K)^{ab}/K)$, and the image of the continuous homomorphism defined by the composition

$$U_{K_v} \hookrightarrow \mathbb{J}_K \xrightarrow{(\bullet, L/K)} \mathrm{Gal}((L/K)^{ab}/K)$$

is the inertia group $\mathrm{I}_v((L/K)^{ab}/K)$ of v in $\mathrm{Gal}((L/K)^{ab}/K)$. Moreover, for each prime v, any π_v is mapped to an element in the Frobenius coset modulo $\mathrm{I}_v((L/K)^{ab}/K)$ in $\mathrm{D}_v((L/K)^{ab}/K)$.

(iii) The triangle

$$\begin{array}{ccc} \mathbb{J}_K & \xrightarrow{(\bullet, E/K)} & \mathrm{Gal}((E/K)^{ab}/K) \\ & {\scriptstyle (\bullet, L/K)} \searrow & \downarrow{\scriptstyle \mathrm{res}_{(L/K)^{ab}}} \\ & & \mathrm{Gal}((L/K)^{ab}/K) \end{array}$$

is commutative, where the right vertical arrow $\mathrm{res}_{(L/K)^{ab}}$ is the "restriction to $(L/K)^{ab}$" map .

(iv) The square

$$\begin{array}{ccc} \mathbb{J}_L & \xrightarrow{(\bullet,E/L)} & \mathrm{Gal}((E/L)^{ab}/L) \\ {\scriptstyle \mathrm{N}_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{(E/K)^{ab}}} \\ \mathbb{J}_K & \xrightarrow{(\bullet,E/K)} & \mathrm{Gal}((E/K)^{ab}/K) \end{array}$$

where the left vertical arrow $\mathrm{N}_{L/K}$ is the idèlic norm map from L to K, and the square

$$\begin{array}{ccc} \mathbb{J}_K & \xrightarrow{(\bullet,E/K)} & \mathrm{Gal}((E/K)^{ab}/K) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Ver}_{K\to L}} \\ \mathbb{J}_L & \xrightarrow{(\bullet,E/L)} & \mathrm{Gal}((E/L)^{ab}/L) \end{array}$$

where the left vertical arrow is the natural inclusion and the right vertical arrow is the “Verlagerung” (transfer) map from K to L (more precisely from G_K to G_L), are commutative.

(v) (Global existence theorem). For any open subgroup N of finite index in $K^\times\backslash\mathbb{J}_K$, there is a unique abelian extension $L_N = L$ over K such that the kernel of the abelian reciprocity homomorphism

$$(\bullet, L/K) : K^\times\backslash\mathbb{J}_K \to \mathrm{Gal}(L/K)$$

relative to the extension L/K is N. Moreover, this assignement

$$N \mapsto L_N$$

defines an inclusion-reversing bijective correspondence

$$\left\{\begin{array}{c}\text{Open subgroups of finite}\\ \text{index in } \mathbb{J}_K \text{ containing } K^\times\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{Finite abelian exten-}\\ \text{sions of } K \text{ inside } K^{sep}\end{array}\right\}.$$

(vi) (Ray class groups and ray class fields). Let $\mathfrak{m} = \prod_{v\in\mathfrak{h}_K\cup\mathfrak{a}_K} v^{e_v}$ be a fixed cycle (i.e., modulus) of the global field K, and let $\mathbb{U}_\mathfrak{m}$ be the subgroup of $\mathbb{J}_K$ defined by $\mathfrak{m}$. Then under the canonical surjective homomorphism $\mathbb{J}_K \twoheadrightarrow K^\times\backslash\mathbb{J}_K$, the subgroup $\mathbb{U}_\mathfrak{m}$ of $\mathbb{J}_K$ maps onto a finite-index open subgroup $\overline{\mathbb{U}}_\mathfrak{m}$ of $K^\times\backslash\mathbb{J}_K$. By the global existence theorem, there exists a finite abelian extension $R_\mathfrak{m}$ of K, called the ray class field of $\mathfrak{m}$, such that the abelian reciprocity homomorphism $(\bullet, R_\mathfrak{m}/K) : \mathbb{J}_K \to \mathrm{Gal}(R_\mathfrak{m}/K)$ relative to the extension $R_\mathfrak{m}/K$ induces an isomorphism

$$\mathbb{U}_\mathfrak{m}K^\times\backslash\mathbb{J}_K \xrightarrow{\sim} \mathrm{Gal}(R_\mathfrak{m}/K),$$

where $\mathbb{U}_\mathfrak{m}K^\times\backslash\mathbb{J}_K$ is called the ray class group of $\mathfrak{m}$.

(vii) (Splitting of primes). A prime v in K splits completely in the abelian extension $(L/K)^{ab}/K$ if and only if $K_v^\times \subset \ker(\bullet, L/K)$. Thus,

$$\mathrm{Spl}\left((L/K)^{ab}/K\right) = \{v \in \mathbb{h}_K : K_v^\times \subset \ker(\bullet, L/K)\},$$

where $\mathrm{Spl}\left((L/K)^{ab}/K\right)$ denotes the set of all primes in K that split completely in the abelian extension $(L/K)^{ab}/K$[4].

1.2. Aim

In [4], we have introduced a certain topological group $\mathscr{I}_K^{\varphi}$ depending only on the global field K and called the *non-abelian idèle group of* K together with a natural continuous homomorphism

$$\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to W_K$$

called the *global non-abelian norm-residue symbol of* K, or the *non-abelian global reciprocity law of* K. Moreover, in [4], we have studied the ℓ-adic representations of the topological group $\mathscr{I}_K^{\varphi_K}$, and observed that the theory of n-dimensional ℓ-adic representations of $\mathscr{I}_K^{\varphi_K}$ is closely related with the Langlands reciprocity principle for $\mathrm{GL}(n)$ over K, for each $n \geq 1$, via the arrow $\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to W_K$.

The aim here, which complements [4], is to study the *basic properties* of the non-abelian global reciprocity law $\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}$ of K and observe that this arrow indeed deserves its name. That is, we shall introduce the non-abelian analogues of the "functoriality" and the "naturality" properties of the abelian global reciprocity law $(\bullet, K)$ of K summarized in 1.1, and then prove these properties. Moreover, we shall describe the set of primes in K that split in a finite extension L of K, which is one of the main goals of non-abelian global class field theory.

1.3. Outline

The outline of this paper is as follows. In Sections 2 and 3, we shall briefly review the theory developed in [4]. More precisely, we shall recall the construction of the non-abelian idèle group $\mathscr{I}_K^{\varphi_K}$ of K and then recall the construction of the non-abelian global reciprocity law $\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to W_K$ of K. In Section 4, we shall state and prove the functoriality and naturality properties of the non-abelian global reciprocity law $\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to W_K$ of the global field K, which constitute the non-abelian analogues of the basic properties of the abelian global reciprocity law $(\bullet, K) : \mathbb{J}_K \to W_K^{ab}$ of K sketched in 1.1. Finally, we end Section 4 by describing the set of primes in K that split in a finite extension L of K.

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[4]By Čebotarev density theorem, $\mathrm{Spl} : L/K \mapsto \mathrm{Spl}(L/K)$ is an injective and order-reversing mapping from finite Galois extensions L of the global field K into the power set of $\mathrm{h}_K \cup \mathbb{O}_K$. The image of the map "Spl" for finite abelian extensions L of K has a description in terms of the abelian global reciprocity law $(\bullet, L/K)$ relative to the extension L/K.

2. Non-abelian idèle group $\mathscr{J}_K^{\varphi}$ of a global field K

All through this work K denotes a global field. That is, K is a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_q(T)$ (that is, the field of rational functions of a curve defined over a finite field $\mathbb{F}_q$). For the basic theory of local fields and the abelian local class field theory, and for details about global fields and the abelian global class field theory, we refer the reader to [10]. Let $\mathbb{a}_K$ denote the set of all archimedean primes of K (so in case K is a function field, then $\mathbb{a}_K = \varnothing$). For each $v \in \mathbb{h}_K$, where $\mathbb{h}_K$ denotes the set of all henselian (=non-archimedean) primes of K, let K_v denote the completion of K with respect to the v-adic absolute value.

Fixing a Lubin-Tate splitting φ_{K_v} over K_v, the non-abelian local reciprocity law

$$\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : G_{K_v} \xrightarrow{\sim} \nabla_{K_v}^{(\varphi_{K_v})}$$

or equivalently the "Weil form" of the non-abelian local reciprocity law

$$\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$$

of the local field K_v is defined. The construction of the topological group $\nabla_{K_v}^{(\varphi_{K_v})}$, which depends on K_v and on the choice of the Lubin-Tate splitting φ_{K_v} over K_v, introduced here involves the theory of *APF*-extensions of K_v and the fields of norms construction of Fontaine and Wintenberger, and ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ is a certain dense subgroup of the topological group $\nabla_{K_v}^{(\varphi_{K_v})}$ (for details, see [6]). Moreover, the isomorphism $\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}$, which is called the *non-abelian local reciprocity law of* K_v, is "natural" in the sense that properties such as "existence", "functoriality" and a certain "ramification theoretic" property are all satisfied. The isomorphism $\{\bullet, K_v\}_{\varphi_{K_v}}$, which is defined to be the inverse $\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})^{-1}}$ of the isomorphism $\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}$ by

$$\{\bullet, K_v\}_{\varphi_{K_v}} : \nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\sim} G_{K_v}$$

is called the *non-abelian local norm-residue symbol of* K_v. For details on non-abelian local class field theory in the sense of Koch, we refer the reader to the papers [5, 6, 7] as well as Laubie's work [9]. Moreover, following Section 8 of [5] together with [7] for a detailed account, for each $v \in \mathbb{h}_K$, there exists the subgroup ${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}$ of ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ satisfying the equality

$$\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}\left(W_{K_v}^{0}\right) = {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}. \tag{2.1}$$

It is then a natural attempt to construct the non-abelian version of global class field theory of the global field K by "glueing" the non-abelian local class field theories of respective completions K_v of K, for $v \in \mathbb{h}_K$, following Chevalley-Miyake philosophy of idèles. This program has been carried out in [4] yielding the non-abelian global reciprocity law of K and the "ultimate" non-abelian global reciprocity law of K by pushing the idea of Miyake introduced in [11, 12] to the extreme, and therefore introducing the non-abelian idèle group $\mathscr{J}_K^{\varphi_K}$ of K by following the analogy between the non-abelian local class field theory in the sense of Koch and the abelian local class field theory of Hasse and taking into account the anology between the philosophy of Miyake and the philosophy of Chevalley (also look at Iwasawa [8]).

2.1. Digression : Restricted free products of locally compact groups

The main reference that we follow very closely and reproduce here is Section 2 of [4]. Let $\{G_i\}_{i\in I}$ be a collection of locally compact topological groups. For all but finitely many $i \in I$, let O_i be a compact open subgroup of G_i. The finite subset of I consisting of all $i \in I$ for which O_i is not defined is denoted by I_∞. For every finite subset S of I satisfying $I_\infty \subseteq S$, define the topological group

$$G_S := \mathop{*}_{i\notin S} O_i * \left(\mathop{*}_{i\in S} G_i\right)$$

as the free product of the topological groups O_i, for $i \in I - S$, and G_i, for $i \in S$, which exists in the category of topological groups (cf. Morris [13]). Then, the *restricted free product* of the collection $\{G_i\}_{i\in I}$ with respect to the collection $\{O_i\}_{i\in I-I_\infty}$, which is denoted by $*'_{i\in I}(G_i : O_i)$, is defined by the injective limit

$$\mathop{*}_{i\in I}{}'(G_i : O_i) := \varinjlim_S G_S$$

defined over all possible such S, where the connecting morphism

$$\tau_S^T : G_S \to G_T$$

for $S \subseteq T$ is defined naturally by the *"universal mapping property of free products"*[5] (cf. Hilton-Wu [3] and Morris [13]). The topology on $*'_{i\in I}(G_i : O_i)$ is defined by declaring $X \subseteq *'_{i\in I}(G_i : O_i)$ to be open if $X \cap G_S$ is open in G_S for every S. So, endowed with this topology, $*'_{i\in I}(G_i : O_i)$ is a topological group.

The following proposition (which is Proposition 2.1 of [4]) is a direct consequence of the "universal mapping property of free products".

Proposition 2.1. *Let $\{G_i\}_{i\in I}$ be a collection of locally compact topological groups and for all but finitely many $i \in I$ let O_i be a compact open subgroup of G_i. Denote the finite subset of I consisting of all $i \in I$ for which O_i is not defined by I_∞. Assume that, for each $i \in I$, a continuous homomorphism*

$$\phi_i : G_i \to H$$

is given. Then, there exists a unique continuous homomorphism

$$\phi_S : G_S \to H$$

defined for each finite subset S of I satisfying $I_\infty \subseteq S$, and a unique continuous homomorphism

$$\phi = \varinjlim_S \phi_S : *'_{i\in I}(G_i : O_i) \to H$$

satisfying

$$\phi_S = \phi \circ c_S : G_S \xrightarrow{c_S} *'_{i\in I}(G_i : O_i) \xrightarrow{\phi} H,$$

[5] If $\{G_i\}_{i\in I}$ is a collection of topological groups and $*_{i\in I} G_i$ is the free product of this collection together with the canonical embeddings $\iota_{i_o} : G_{i_o} \hookrightarrow *_{i\in I} G_i$, for each $i_o \in I$, then the universal mapping property of free products states that, if for each $i_o \in I$, $\phi_{i_o} : G_{i_o} \to H$ is a continuous homomorphism, then there exists a unique continuous homomorphism $\phi : *_{i\in I} G_i \to H$, such that $\phi \circ \iota_{i_o} = \phi_{i_o}$, for every $i_o \in I$.

where $c_S : G_S \to *'_{i\in I}(G_i : O_i)$ *is the canonical homomorphism, for every* S.

Notation 2.2. As a notation, for a topological group G, the n-fold free product $\overbrace{G*\cdots*G}^{n\text{-copies}}$ of G is denoted by G^{*n}.

2.2. Definition of the non-abelian idèle group $\mathscr{I}_K^{\underline{\varphi}}$ of K

Now, we introduce the non-abelian idèle group $\mathscr{I}_K^{\underline{\varphi}}$ of K as follows.

Definition 2.3. For each $v \in \mathbb{h}_K$ fix a Lubin-Tate splitting φ_{K_v} and let $\underline{\varphi}_K = \{\varphi_{K_v}\}_{v\in\mathbb{h}_K}$. If there is no fear of confusion, denote $\underline{\varphi}_K = \underline{\varphi}$. The topological group $\mathscr{I}_K^{\underline{\varphi}}$ defined by the "restricted free product"

$$\mathscr{I}_K^{\underline{\varphi}} := \underset{v\in\mathbb{h}_K\cup\mathbb{a}_K}{*}{}' \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \right)$$

is called the *non-abelian idèle group of the global field* K. In case K is a number field,

$$\mathscr{I}_K^{\underline{\varphi}} = \underset{v\in\mathbb{h}_K}{*}{}' \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \right) * W_{\mathbb{R}}^{*r_1} * W_{\mathbb{C}}^{*r_2},$$

where the finite (=henselian) part $\mathscr{I}_{K,\mathbb{h}}^{\underline{\varphi}}$ of $\mathscr{I}_K^{\underline{\varphi}}$ is defined by

$$\mathscr{I}_{K,\mathbb{h}}^{\underline{\varphi}} := \underset{v\in\mathbb{h}_K}{*}{}' \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \right),$$

and the infinite (=archimedean) part $\mathscr{I}_{K,\mathbb{a}}^{\underline{\varphi}}$ of $\mathscr{I}_K^{\underline{\varphi}}$ by

$$\mathscr{I}_{K,\mathbb{a}}^{\underline{\varphi}} := W_{\mathbb{R}}^{*r_1} * W_{\mathbb{C}}^{*r_2}.$$

Here, as usual r_1 and r_2 denote the number of real and the number of complex conjugate embeddings of the global field K in $\mathbb{C}$.

Remark 2.4. Note that, the non-abelian idèle group $\mathscr{I}_K^{\underline{\varphi}}$ of K depends only on the global field K and the choice of $\underline{\varphi}$.

Theorem 2.5 of [4] states that, the abelianization $\mathscr{I}_K^{\underline{\varphi}\,ab}$ of the topological group $\mathscr{I}_K^{\underline{\varphi}}$ is canonically isomorphic to the idèle group $\mathbb{J}_K$ of K. Therefore, there exists a continuous surjective homomorphism

$$\mathsf{s}_K : \mathscr{I}_K^{\underline{\varphi}_K} \twoheadrightarrow \mathbb{J}_K$$

defined by the abelianization map

$$\mathscr{I}_K^{\underline{\varphi}_K} \to \mathscr{I}_K^{\underline{\varphi}\,ab}$$

of $\mathscr{I}_K^{\underline{\varphi}_K}$.

For each $v \in \mathbb{h}_K \cup \mathbb{a}_K$, there exists a natural homomorphism

$$q_v : (\mathscr{I}_K^{\underline{\varphi}})_v := \left\{ \begin{matrix} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}, & v \in \mathbb{h}_K \\ W_{\mathbb{R}}, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}, & v \in \mathbb{a}_{K,\mathbb{C}} \end{matrix} \right\} \to \mathscr{I}_K^{\underline{\varphi}},$$

which is defined explicitly via the commutative triangle

$$\begin{array}{ccc} & & (\mathscr{I}_K^{\varphi})_S \\ & \overset{\iota_v^{(S)}}{\nearrow} & \\ (\mathscr{I}_K^{\varphi})_v & & \Big\downarrow c_S \\ & \underset{q_v}{\searrow} & \\ & & \mathscr{I}_K^{\varphi} \end{array}$$

where S is a finite subset of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $\mathbb{a}_K \subseteq S$ and $v \in S$. Note that, the definition of the continuous homomorphism $q_v : (\mathscr{I}_K^{\varphi})_v \to \mathscr{I}_K^{\varphi}$ does not depend on the choice of S (for details look at Section 4 of [4]).

3. Non-abelian global reciprocity law

For $v \in \mathbb{h}_K \cup \mathbb{a}_K$, choose an embedding

$$e_v : K^{sep} \hookrightarrow K_v^{sep}.$$

This embedding determines a continuous homomorphism [6] (look at [14] for details)

$$e_v^{\text{Weil}} : W_{K_v} \to W_K,$$

and, for each $v \in \mathbb{h}_K$, a continuous homomorphism

$$\mathsf{NR}_{K_v}^{(\varphi_{K_v})^{\text{Weil}}} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow[\sim]{\{\bullet, K_v\}_{\varphi_{K_v}}} W_{K_v} \xrightarrow{e_v^{\text{Weil}}} W_K.$$

By the "universal mapping property of free products", the following theorem (which is Theorem 3.1 of [4]) follows.

Theorem 3.1. *There exists a well-defined continuous homomorphism*

$$\mathsf{NR}_K^{\varphi^{\text{Weil}}} : \mathscr{I}_K^{\varphi} \to W_K, \tag{3.1}$$

called the non-abelian global reciprocity law of K, or the global non-abelian norm-residue symbol of K, which satisfies

$$(\mathsf{NR}_K^{\varphi^{\text{Weil}}})_S = \mathsf{NR}_K^{\varphi^{\text{Weil}}} \circ c_S : (\mathscr{I}_K^{\varphi})_S \xrightarrow{c_S} \mathscr{I}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi^{\text{Weil}}}} W_K,$$

where $c_S : (\mathscr{I}_K^{\varphi})_S \to \mathscr{I}_K^{\varphi}$ is the canonical homomorphism defined for every finite subset S of $\mathbb{h}_K \cup \mathbb{a}_K$ containing $\mathbb{a}_K$.

Moreover, we have made the following conjecture (look at Conjecture 3.2 in [4]) :

Conjecture 3.2. *The homomorphism*

$$\mathsf{NR}_K^{\varphi^{\text{Weil}}} : \mathscr{I}_K^{\varphi} \to W_K$$

is open, continuous and surjective.

[6] which is unique if K is a function field and unique up to composition with an inner automorphism of W_K defined by an element of the connected component W_K^o of W_K if K is a number field.

In this work, we shall assume that Conjecture 3.2 holds only in Subsections 4.4, 4.8, 4.9, and 4.10.

4. Basic properties of the non-abelian global reciprocity law

In the remaining of this paper, we shall study the basic properties of the non-abelian global reciprocity law $\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to W_K$ of the global field K.

4.1. Local-global compatibility of the non-abelian norm residue symbols

The *"local-global compatibility"* of $\{\bullet, K_v\}_{\varphi_{K_v}}$ and $\mathsf{NR}_K^{\varphi^{Weil}}$ for $v \in \mathbb{h}_K$, proved in [4] as Theorem 4.1, states the commutativity of the square

$$\begin{array}{ccc} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} & \xrightarrow{q_v} & \mathscr{I}_K^{\varphi} \\ {\scriptstyle \{\bullet, K_v\}_{\varphi_{K_v}}}\downarrow & & \downarrow{\scriptstyle \mathsf{NR}_K^{\varphi^{\mathrm{Weil}}}} \\ W_{K_v} & \xrightarrow[e_v^{\mathrm{Weil}}]{} & W_K \end{array}$$

for each $v \in \mathbb{h}_K$.

4.2. Relationship with the abelian global reciprocity law

In this subsection, we shall study the "behaviour" of the non-abelian global reciprocity law under the abelianization functor, and prove that under abelianization, the non-abelian global reciprocity law reduces to the abelian global reciprocity law.

For $v \in \mathbb{h}_K$, define the surjective and continuous homomorphism $\mathsf{a}_{K_v} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \to K_v^\times$ by the composition

$$\mathsf{a}_{K_v} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\mathrm{id}_{\mathbb{Z}} \times \mathrm{Pr}_{\widetilde{K}_v}} \mathbb{Z} \times U_{K_v} \xrightarrow{\sim} K_v^\times$$

following Remark 4 in [2] combined with the construction of the non-abelian local norm-residue symbol $\{\bullet, K_v\}_{\varphi_{K_v}} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\sim} W_{K_v}$ of K_v in the "Weil form" as described in [6]. For $v \in \mathbb{o}_K$, define the continuous homomorphism

$$\mathsf{a}_{K_v} : W_{K_v} \to K_v^\times$$

to be the natural homomorphism defined by abelianization $W_{K_v}^{ab} \xrightarrow{\sim} K_v^\times$ of W_{K_v}.

Lemma 4.1. *There exists a unique continuous homomorphism*

$$\mathsf{a}_K : \mathscr{I}_K^{\varphi} \to \mathbb{J}_K$$

which makes the diagram

$$\begin{array}{ccc} (\mathscr{I}_K^{\varphi})_v & \xrightarrow{q_v} & \mathscr{I}_K^{\varphi} \\ {\scriptstyle \mathsf{a}_{K_v}}\downarrow & & \downarrow{\scriptstyle \mathsf{a}_K} \\ K_v^\times & \xrightarrow[\varepsilon_v]{} & \mathbb{J}_K \end{array}$$

commutative.

Proof. Follows from Proposition 2.1 of [4] applied to the collection of continuous homomorphisms $\varepsilon_v \circ \mathsf{a}_K : (\mathscr{I}^{\varphi}_{K})_v \xrightarrow{\mathsf{a}_{K_v}} K_v^{\times} \xrightarrow{\varepsilon_v} \mathbb{J}_K$ defined for each $v \in \mathbb{h}_K \cup \mathbb{a}_K$. □

Lemma 4.2. *For each $v \in \mathbb{h}_K \cup \mathbb{a}_K$, there exists a continuous homomorphism*

$$q_v^{ab} : (\mathscr{I}^{\varphi}_{K})_v^{ab} \to \mathscr{I}^{\varphi\, ab}_{K},$$

called the abelianization of the natural homomorphism $q_v : (\mathscr{I}^{\varphi}_{K})_v \to \mathscr{I}^{\varphi}_{K}$, that makes the diagram

$$\begin{array}{ccc} (\mathscr{I}^{\varphi}_{K})_v & \xrightarrow{q_v} & \mathscr{I}^{\varphi}_{K} \\ {\scriptstyle \mathrm{red}_{(\mathscr{I}^{\varphi}_{K})_v{}^{c}}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_{\mathscr{I}^{\varphi\, c}_{K}}} \\ (\mathscr{I}^{\varphi}_{K})_v^{ab} & \xrightarrow[q_v^{ab}]{} & \mathscr{I}^{\varphi\, ab}_{K} \end{array}$$

commutative.

Proof. The proof is trivial. □

Theorem 4.3. *The non-abelian global reciprocity law*

$$\mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K} : \mathscr{I}^{\varphi}_{K} \to W_K$$

of K sits in the following commutative diagram

$$\begin{array}{ccc} \mathscr{I}^{\varphi}_{K} & \xrightarrow{\mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K}} & W_K \\ {\scriptstyle \mathsf{a}_K}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_{W_K^c}} \\ \mathbb{J}_K & \xrightarrow[(\bullet,K)]{} & W_K^{ab}. \end{array}$$

Proof. It suffices to prove the equality

$$(\mathrm{red}_{W_K^c} \circ \mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K})_S \circ \iota_v^{(S)} = ((\bullet, K) \circ \mathsf{a}_K)_S \circ \iota_v^{(S)},$$

where S is any finite subset of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $\mathbb{a}_K \subseteq S$ and $v \in S$. In fact, for such an S and for any $v \in S$, in case $v \in \mathbb{h}_K$, then

$$\begin{aligned} (\mathrm{red}_{W_K^c} \circ \mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K})_S \circ \iota_v^{(S)} &= \mathrm{red}_{W_K^c} \circ \mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K} \circ c_S \circ \iota_v^{(S)} \\ &= \mathrm{red}_{W_K^c} \circ (\mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K} \circ q_v) \\ &= \mathrm{red}_{W_K^c} \circ (e_v^{\mathrm{Weil}} \circ \{\bullet, K_v\}_{\varphi_v}). \end{aligned}$$

Moreover, the following diagram

$$\begin{array}{ccccc} {}_{\mathbb{Z}}\nabla^{(\varphi_{K_v})}_{K_v} & \xrightarrow[\sim]{\{\bullet, K_v\}_{\varphi_{K_v}}} & W_{K_v} & \xrightarrow{e_v^{\mathrm{Weil}}} & W_K \\ {\scriptstyle \mathsf{a}_{K_v}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_{W^c_{K_v}}} & & \downarrow{\scriptstyle \mathrm{red}_{W_K^c}} \\ K_v^{\times} & \xrightarrow[(\bullet, K_v)]{\sim} & W_{K_v}^{ab} & \xrightarrow[e_v^{\mathrm{Weil}^{ab}}]{} & W_K^{ab} \end{array} \tag{4.1}$$

commutes, as the rectangle

$$\begin{array}{ccc} W_{K_v} & \xrightarrow{e_v^{\mathrm{Weil}}} & W_K \\ {\scriptstyle \mathrm{red}_{W_{K_v}^c}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_{W_K^c}} \\ W_{K_v}^{ab} & \xrightarrow[e_v^{\mathrm{Weil}^{ab}}]{} & W_K^{ab} \end{array}$$

is naturally commutative, and the diagram

$$\begin{array}{ccc} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} & \xrightarrow[\sim]{\{\bullet,K_v\}_{\varphi_{K_v}}} & W_{K_v} \\ {\scriptstyle \mathsf{a}_{K_v}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_{W_{K_v}^c}} \\ K_v^{\times} & \xrightarrow[(\bullet,K_v)]{\sim} & W_{K_v}^{ab} \end{array}$$

which relates the abelian with the non-abelian local reciprocity laws is commutative by Remark 4 in [2] combined with the construction of the non-abelian local norm-residue symbol $\{\bullet,K_v\}_{\varphi_{K_v}} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\sim} W_{K_v}$ of K_v in the "Weil form" as described in [6], Therefore,

$$\begin{aligned} (\mathrm{red}_{W_K^c} \circ \mathsf{NR}_K^{\underline{\varphi}^{\mathrm{Weil}}})_S \circ \iota_v^{(S)} &= \mathrm{red}_{W_K^c} \circ (e_v^{\mathrm{Weil}} \circ \{\bullet, K_v\}_{\varphi_v}) \\ &= (e_v^{\mathrm{Weil}^{ab}} \circ (\bullet, K_v)) \circ \mathsf{a}_{K_v} \\ &= ((\bullet, K) \circ \varepsilon_v) \circ \mathsf{a}_{K_v}, \end{aligned}$$

by the commutativity of the diagram (4.1) and by the compatibility diagram (1.1) of the local and the global abelian reciprocity laws. Now, by Lemma 4.1,

$$\begin{aligned} (\mathrm{red}_{W_K^c} \circ \mathsf{NR}_K^{\underline{\varphi}^{\mathrm{Weil}}})_S \circ \iota_v^{(S)} &= ((\bullet, K) \circ \varepsilon_v) \circ \mathsf{a}_{K_v} \\ &= (\bullet, K) \circ (\mathsf{a}_K \circ q_v) \\ &= (\bullet, K) \circ \mathsf{a}_K \circ c_S \circ \iota_v^{(S)} \\ &= ((\bullet, K) \circ \mathsf{a}_K)_S \circ \iota_v^{(S)}, \end{aligned}$$

which completes the proof. The case $v \in \mathbb{O}_K$ can be proved similarly. □

4.3. Non-abelian idèles in field extensions

In this subsection, we shall study the relationship between the non-abelian idèle group $\mathscr{I}_K^{\underline{\varphi}_K}$ of K and the non-abelian idèle group $\mathscr{I}_L^{\underline{\varphi}_L}$ of L, where L is a finite extension of the global field K.

Remark 4.4. Let L be a finite extension of the global field K. Fixing $\underline{\varphi}_K = \{\varphi_{K_v}\}_{v \in \mathbb{h}_K}$ uniquely determines $\underline{\varphi}_L = \{\varphi_{L_\mu}\}_{\mu \in \mathbb{h}_L}$ via Koch-de Shalit process applied to compatible extensions of K_v for each $v \in \mathbb{h}_K$ (for details, look at [6]).

Thus, let L be a finite extension of the global field K. The absolute Weil group W_L of L is the open subgroup of W_K defined by $W_L = \beta_K^{-1}(G_L)$, where the absolute

Weil group W_K of the global field K comes equipped with a continuous homomorphism $\beta_K : W_K \to G_K$ with dense image. Moreover, the open subgroup W_L of W_K is equipped with a continuous homomorphism $\beta_L : W_L \to G_L$, which sits in the commutative square

$$\begin{array}{ccc} W_L & \xrightarrow{\beta_L} & G_L \\ {\scriptstyle \gamma_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_L}} \\ W_K & \xrightarrow[\beta_K]{} & G_K \end{array} \tag{4.2}$$

and with dense image, where the left vertical arrow

$$\gamma_{L/K} : W_L \hookrightarrow W_K$$

is the natural embedding; that is, the identity map defined by the inclusion mapping $W_L := \beta_K^{-1}(G_L) \subset W_K$. Let $v \in \mathbb{h}_K \cup \mathbb{a}_K$ and $\mu \in \mathbb{h}_L \cup \mathbb{a}_L$ so that $\mu \mid v$. Then, L_μ is a finite extension of K_v, and W_{L_μ} is an open subgroup of W_{K_v} defined by $W_{L_\mu} = \beta_{K_v}^{-1}(G_{L_\mu})$, where the absolute Weil group W_{K_v} of the local field K_v comes equipped with a continuous homomorphism $\beta_{K_v} : W_{K_v} \to G_{K_v}$ with dense image. The open subgroup W_{L_μ} of W_{K_v} is equipped with a continuous homomorphism $\beta_{L_\mu} : W_{L_\mu} \to G_{L_\mu}$ with dense image, and the square

$$\begin{array}{ccc} W_{L_\mu} & \xrightarrow{\beta_{L_\mu}} & G_{L_\mu} \\ {\scriptstyle \gamma_{L_\mu/K_v}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_{L_\mu}}} \\ W_{K_v} & \xrightarrow[\beta_{K_v}]{} & G_{K_v} \end{array} \tag{4.3}$$

commutes, where the left vertical arrow

$$\gamma_{L_\mu/K_v} : W_{L_\mu} \hookrightarrow W_{K_v}$$

is the natural embedding; namely, the identity mapping defined by the inclusion $W_{L_\mu} := \beta_{K_v}^{-1}(G_{L_\mu}) \subset W_{K_v}$.

For any place $v \in \mathbb{h}_K \cup \mathbb{a}_K$ of K, the fixed embedding

$$e_v : K^{sep} \hookrightarrow K_v^{sep} \tag{4.4}$$

uniquely determines a continuous homomorphism

$$e_v^{\mathrm{Galois}} : G_{K_v} \to G_K$$

defined by the restriction to K^{sep} and a continuous homomorphism

$$e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$$

so that the following square

$$\begin{array}{ccc} W_{K_v} & \xrightarrow{e_v^{\mathrm{Weil}}} & W_K \\ {\scriptstyle \beta_{K_v}}\downarrow & & \downarrow{\scriptstyle \beta_K} \\ G_{K_v} & \xrightarrow[e_v^{\mathrm{Galois}}]{} & G_K \end{array} \tag{4.5}$$

is commutative (for details, look at Proposition 1.6.1 of Tate [14]). Moreover, for any finite extension L/K, the fixed embedding (4.4) uniquely determines an embedding

$$e_\mu : L^{sep} \hookrightarrow L_\mu^{sep} \tag{4.6}$$

which is defined to be the unique arrow that makes the square

$$\begin{array}{ccc} L^{sep} & \overset{\exists! e_\mu}{\hookrightarrow} & L_\mu^{sep} \\ \| & & \| \\ K^{sep} & \underset{e_\nu}{\hookrightarrow} & K_\nu^{sep} \end{array}$$

commutative, for every $\mu \in \mathbb{h}_L \cup \mathbb{a}_L$ satisfying $\mu \mid \nu$. Therefore, the embedding (4.6) uniquely determines a continuous homomorphism

$$e_\mu^{\text{Galois}} : G_{L_\mu} \to G_L$$

defined by the restriction to L^{sep}. The relationship between the continuous homomorphisms e_ν^{Galois} and e_μ^{Galois} is given by the following lemma.

Lemma 4.5. *Let L be a finite extension of the global field K. For any finite or infinite prime ν of K and for any prime μ of L lying above ν, the following square*

$$\begin{array}{ccc} G_{L_\mu} & \xrightarrow{e_\mu^{\text{Galois}}} & G_L \\ {\scriptstyle \text{id}_{G_{L_\mu}}}\downarrow & & \downarrow{\scriptstyle \text{id}_{G_L}} \\ G_{K_\nu} & \xrightarrow[e_\nu^{\text{Galois}}]{} & G_K \end{array} \tag{4.7}$$

is commutative.

Proof. In fact, for any $\sigma \in G_{L_\mu}$,

$$\begin{aligned} \text{id}_{G_L} \circ e_\mu^{\text{Galois}}(\sigma) &= \sigma\,|_{L^{sep}} \\ &= \sigma\,|_{K^{sep}} \\ &= e_\nu^{\text{Galois}} \circ \text{id}_{G_{L_\mu}}(\sigma) \end{aligned}$$

which completes the proof of the commutativity of the square (4.7). □

The embedding (4.6) also determines a continuous homomorphism

$$e_\mu^{\text{Weil}} : W_{L_\mu} \to W_L$$

defined by

$$e_\mu^{\text{Weil}}(w) = e_\nu^{\text{Weil}} \circ \gamma_{L_\mu/K_\nu}(w),$$

for every $w \in W_{L_\mu}$. In fact, for each $w \in W_{L_\mu}$, the commutativity of the square (4.5) yields

$$\begin{aligned} (\beta_K \circ e_\nu^{\text{Weil}}) \circ \gamma_{L_\mu/K_\nu}(w) &= (e_\nu^{\text{Galois}} \circ \beta_{K_\nu}) \circ \gamma_{L_\mu/K_\nu}(w) \\ &= e_\nu^{\text{Galois}} \circ (\beta_{K_\nu} \circ \gamma_{L_\mu/K_\nu})(w) \\ &= e_\nu^{\text{Galois}} \circ (\text{id}_{G_{L_\mu}} \circ \beta_{L_\mu})(w) \end{aligned}$$

where the last equality follows from the commutative square (4.3). Therefore,

$$\begin{aligned}(\beta_K \circ e_\nu^{\mathrm{Weil}}) \circ \gamma_{L_\mu/K_\nu}(w) &= e_\nu^{\mathrm{Galois}} \circ (\mathrm{id}_{G_{L_\mu}} \circ \beta_{L_\mu})(w)\\ &= (e_\nu^{\mathrm{Galois}} \circ \mathrm{id}_{G_{L_\mu}}) \circ \beta_{L_\mu}(w)\\ &= (\mathrm{id}_{G_L} \circ e_\mu^{\mathrm{Galois}}) \circ \beta_{L_\mu}(w)\end{aligned}$$

by Lemma 4.5. Therefore, for each $w \in W_{L_\mu}$,

$$e_\nu^{\mathrm{Weil}} \circ \gamma_{L_\mu/K_\nu}(w) \in \beta_K^{-1}(G_L) = W_L.$$

The relationship between the continuous homomorphisms e_ν^{Weil} and e_μ^{Weil} is described in the following lemma.

Lemma 4.6. *Let L be a finite extension of the global field K. For any finite or infinite prime ν of K and for any prime μ of L lying above ν, the following square*

$$\begin{array}{ccc} W_{L_\mu} & \xrightarrow{e_\mu^{\mathrm{Weil}}} & W_L \\ {\scriptstyle \gamma_{L_\mu/K_\nu}}\downarrow & & \downarrow{\scriptstyle \gamma_{L/K}} \\ W_{K_\nu} & \xrightarrow[e_\nu^{\mathrm{Weil}}]{} & W_K \end{array} \tag{4.8}$$

is commutative.

Proof. For any $w \in W_{L_\mu}$, the following equalities

$$\begin{aligned}e_\nu^{\mathrm{Weil}} \circ \gamma_{L_\mu/K_\nu}(w) &= e_\mu^{\mathrm{Weil}}(w)\\ &= \gamma_{L/K} \circ e_\mu^{\mathrm{Weil}}(w)\end{aligned}$$

are immediate from the definition of $e_\mu^{\mathrm{Weil}} : W_{L_\mu} \to W_L$, completing the proof of the commutativity of the square (4.8). □

Moreover, the continuous homomorphisms $e_\mu^{\mathrm{Galois}} : G_{L_\mu} \to G_L$ and $e_\mu^{\mathrm{Weil}} : W_{L_\mu} \to W_L$ make the following square

$$\begin{array}{ccc} W_{L_\mu} & \xrightarrow{e_\mu^{\mathrm{Weil}}} & W_L \\ {\scriptstyle \beta_{L_\mu}}\downarrow & & \downarrow{\scriptstyle \beta_L} \\ G_{L_\mu} & \xrightarrow[e_\mu^{\mathrm{Galois}}]{} & G_L \end{array} \tag{4.9}$$

commutative. In fact,

$$\begin{aligned}\beta_L \circ e_\mu^{\mathrm{Weil}} &= (\beta_K \circ \gamma_{L/K}) \circ e_\mu^{\mathrm{Weil}}\\ &= \beta_K \circ e_\nu^{\mathrm{Weil}} \circ \gamma_{L_\mu/K_\nu},\end{aligned}$$

where the equalities follow from the commutativity of the squares (4.2) and (4.8). Therefore,

$$\begin{aligned}\beta_L \circ e_\mu^{\mathrm{Weil}} &= (\beta_K \circ e_\nu^{\mathrm{Weil}}) \circ \gamma_{L_\mu/K_\nu} \\ &= e_\nu^{\mathrm{Galois}} \circ (\beta_{K_\nu} \circ \gamma_{L_\mu/K_\nu}) \\ &= e_\nu^{\mathrm{Galois}} \circ (\mathrm{id}_{G_{L_\mu}} \circ \beta_{L_\mu})\end{aligned}$$

by the commutative squares (4.5) and (4.3). Thus, by Lemma 4.5,

$$\begin{aligned}\beta_L \circ e_\mu^{\mathrm{Weil}} &= (e_\nu^{\mathrm{Galois}} \circ \mathrm{id}_{G_{L_\mu}}) \circ \beta_{L_\mu} \\ &= \mathrm{id}_{G_L} \circ (e_\mu^{\mathrm{Galois}} \circ \beta_{L_\mu}) \\ &= e_\mu^{\mathrm{Galois}} \circ \beta_{L_\mu},\end{aligned}$$

which completes the proof of the commutativity of the square (4.9).

Now, for any $\nu \in \mathbb{h}_K$ and for any $\mu \in \mathbb{h}_L$ satisfying $\mu \mid \nu$, the following square

$$\begin{array}{ccc} W_{L_\mu} & \xrightarrow[\sim]{\mathbf{\Phi}_{L_\mu}^{(\varphi_{L_\mu})}} & {}_{\mathbb{Z}}\nabla_{L_\mu}^{(\varphi_{L_\mu})} \\ {\scriptstyle \gamma_{L_\mu/K_\nu}}\downarrow & & \downarrow{\scriptstyle \mathscr{N}_{L_\mu/K_\nu}^{\infty}} \\ W_{K_\nu} & \xrightarrow[\sim]{\mathbf{\Phi}_{K_\nu}^{(\varphi_{K_\nu})}} & {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \end{array} \tag{4.10}$$

is commutative (look at pp. 39 of [6] for details), where the right vertical arrow

$$\mathscr{N}_{L_\mu/K_\nu}^{\infty} : {}_{\mathbb{Z}}\nabla_{L_\mu}^{(\varphi_{L_\mu})} \to {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}$$

is the continuous homomorphism defined in pp. 39 of [6].

For each $\mu \in \mathbb{h}_L$, define a continuous homomorphism

$${}_{\mathbb{Z}}\nabla_{L_\mu}^{(\varphi_{L_\mu})} \to \mathscr{J}_K^{\varphi_K}$$

by the composition

$${}_{\mathbb{Z}}\nabla_{L_\mu}^{(\varphi_{L_\mu})} \xrightarrow{\mathscr{N}_{L_\mu/K_\nu}^{\infty}} {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \xrightarrow{q_\nu} \mathscr{J}_K^{\varphi_K},$$

and for each $\mu \in \mathbb{a}_L$, define a continuous homomorphism

$$W_{L_\mu} \to \mathscr{J}_K^{\varphi_K}$$

by the composition

$$W_{L_\mu} \underset{\gamma_{L_\mu/K_\nu}}{\hookrightarrow} W_{K_\nu} \xrightarrow{q_\nu} \mathscr{J}_K^{\varphi_K}.$$

By Proposition 2.1 of [4], the following proposition follows at once.

Proposition 4.7. *There exists a unique continuous homomorphism*

$$\mathscr{N}_{L/K}^{\infty} : \mathscr{J}_L^{\varphi_L} \to \mathscr{J}_K^{\varphi_K},$$

called the "norm" homomorphism between the non-abelian idèle group $\mathscr{J}_L^{\varphi_L}$ *of L and the non-abelian idèle group* $\mathscr{J}_K^{\varphi_K}$ *of K, which satisfies*

$$(\mathscr{N}_{L/K}^{\infty})_S = \mathscr{N}_{L/K}^{\infty} \circ c_S : (\mathscr{J}_L^{\varphi_L})_S \xrightarrow{c_S} \mathscr{J}_L^{\varphi_L} \xrightarrow{\mathscr{N}_{L/K}^{\infty}} \mathscr{J}_K^{\varphi_K},$$

where $c_S : (\mathscr{J}_L^{\varphi_L})_S \to \mathscr{J}_L^{\varphi_L}$ *is the canonical homomorphism defined for every finite subset S of* $\mathbb{h}_L \cup \mathbb{a}_L$ *containing* $\mathbb{a}_L$*.*

Note that, this homomorphism is transitive in the tower of finite extensions of the global field K.

Proposition 4.8. *Let* $K \subseteq L \subseteq E$ *be a tower of finite extensions of the global field* K*. Then, the equality*

$$\mathscr{N}_{E/K}^{\infty} = \mathscr{N}_{L/K}^{\infty} \circ \mathscr{N}_{E/L}^{\infty}$$

holds.

Proof. It suffices to prove that

$$(\mathscr{N}_{E/K}^{\infty})_S \circ \iota_{\lambda}^{(S)} = (\mathscr{N}_{L/K}^{\infty} \circ \mathscr{N}_{E/L}^{\infty})_S \circ \iota_{\lambda}^{(S)},$$

where $\lambda \in \mathbb{h}_E \cup \mathbb{a}_E$ and S a finite subset of $\mathbb{h}_E \cup \mathbb{a}_E$ such that $\lambda \in S$ and $\mathbb{a}_E \subset S$. So, let λ be any prime of E and S any such subset of $\mathbb{h}_E \cup \mathbb{a}_E$. The composite homomorphism

$$(\mathscr{N}_{E/K}^{\infty})_S = \mathscr{N}_{E/K}^{\infty} \circ c_S : (\mathscr{J}_E^{\varphi_E})_S \xrightarrow{c_S} \mathscr{J}_E^{\varphi_E} \xrightarrow{\mathscr{N}_{E/K}^{\infty}} \mathscr{J}_K^{\varphi_K}$$

satisfies

$$(\mathscr{N}_{E/K}^{\infty})_S \circ \iota_{\lambda}^{(S)} = (\mathscr{N}_{E/K}^{\infty})_{\lambda}$$

and

$$(\mathscr{N}_{E/K}^{\infty})_S \circ \iota_{\lambda}^{(S)} = (\mathscr{N}_{E/K}^{\infty} \circ c_S) \circ \iota_{\lambda}^{(S)} = \mathscr{N}_{E/K}^{\infty} \circ (c_S \circ \iota_{\lambda}^{(S)}) = \mathscr{N}_{E/K}^{\infty} \circ q_{\lambda}.$$

Therefore,

$$(\mathscr{N}_{E/K}^{\infty})_S \circ \iota_{\lambda}^{(S)} = \mathscr{N}_{E/K}^{\infty} \circ q_{\lambda} = (\mathscr{N}_{E/K}^{\infty})_{\lambda}.$$

If $\lambda \in \mathbb{h}_E$, then

$$(\mathscr{N}_{E/K}^{\infty})_S \circ \iota_{\lambda}^{(S)} = \mathscr{N}_{E/K}^{\infty} \circ q_{\lambda} = (\mathscr{N}_{E/K}^{\infty})_{\lambda} = q_{\nu} \circ \mathscr{N}_{E_{\lambda}/K_{\nu}}^{\infty},$$

where $\nu \in \mathbb{h}_K$ is the finite prime of K given by $\nu = \lambda \cap O_K$. Moreover,

$$\begin{aligned}(\mathscr{N}_{L/K}^{\infty} \circ \mathscr{N}_{E/L}^{\infty})_S \circ \iota_{\lambda}^{(S)} &= (\mathscr{N}_{L/K}^{\infty} \circ \mathscr{N}_{E/L}^{\infty}) \circ q_{\lambda} \\ &= \mathscr{N}_{L/K}^{\infty} \circ (\mathscr{N}_{E/L}^{\infty} \circ q_{\lambda}) \\ &= \mathscr{N}_{L/K}^{\infty} \circ (\mathscr{N}_{E/L}^{\infty})_{\lambda} \\ &= \mathscr{N}_{L/K}^{\infty} \circ (q_{\mu} \circ \mathscr{N}_{E_{\lambda}/L_{\mu}}^{\infty}),\end{aligned}$$

where $\mu \in \mathbb{h}_L$ is the finite prime of L defined by $\mu = \lambda \cap O_L$. So it follows that

$$\begin{aligned}(\mathcal{N}^{\infty}_{L/K} \circ \mathcal{N}^{\infty}_{E/L})_S \circ \iota^{(S)}_{\lambda} &= \mathcal{N}^{\infty}_{L/K} \circ (q_\mu \circ \mathcal{N}^{\infty}_{E_\lambda/L_\mu}) \\ &= (\mathcal{N}^{\infty}_{L/K} \circ q_\mu) \circ \mathcal{N}^{\infty}_{E_\lambda/L_\mu} \\ &= (\mathcal{N}^{\infty}_{L/K})_\mu \circ \mathcal{N}^{\infty}_{E_\lambda/L_\mu} \\ &= q_\nu \circ \mathcal{N}^{\infty}_{L_\mu/K_\nu} \circ \mathcal{N}^{\infty}_{E_\lambda/L_\mu},\end{aligned}$$

where $\nu \in \mathbb{h}_K$ is the finite prime of K given by $\nu = \lambda \cap O_K = \mu \cap O_K$. Now, by the transitivity rule proved in pp. 39 of [6],

$$\mathcal{N}^{\infty}_{L_\mu/K_\nu} \circ \mathcal{N}^{\infty}_{E_\lambda/L_\mu} = \mathcal{N}^{\infty}_{E_\lambda/K_\nu}.$$

Therefore, it follows that

$$(\mathcal{N}^{\infty}_{E/K})_S \circ \iota^{(S)}_{\lambda} = (\mathcal{N}^{\infty}_{L/K} \circ \mathcal{N}^{\infty}_{E/L})_S \circ \iota^{(S)}_{\lambda}$$

for $\lambda \in \mathbb{h}_E$. If $\lambda \in \mathbb{a}_E$, then

$$(\mathcal{N}^{\infty}_{E/K})_S \circ \iota^{(S)}_{\lambda} = \mathcal{N}^{\infty}_{E/K} \circ q_\lambda = (\mathcal{N}^{\infty}_{E/K})_\lambda = q_\nu \circ \gamma_{E_\lambda/K_\nu},$$

where $\nu \in \mathbb{a}_K$ is the infinite prime of K defined by $\nu = \lambda \mid_K$. Also,

$$\begin{aligned}(\mathcal{N}^{\infty}_{L/K} \circ \mathcal{N}^{\infty}_{E/L})_S \circ \iota^{(S)}_{\lambda} &= (\mathcal{N}^{\infty}_{L/K} \circ \mathcal{N}^{\infty}_{E/L}) \circ q_\lambda \\ &= \mathcal{N}^{\infty}_{L/K} \circ (\mathcal{N}^{\infty}_{E/L} \circ q_\lambda) \\ &= \mathcal{N}^{\infty}_{L/K} \circ (\mathcal{N}^{\infty}_{E/L})_\lambda \\ &= \mathcal{N}^{\infty}_{L/K} \circ (q_\mu \circ \gamma_{E_\lambda/L_\mu}),\end{aligned}$$

where $\mu \in \mathbb{a}_L$ is the infinite prime of L defined by $\mu = \lambda \mid_L$. Thus, it follows that

$$\begin{aligned}(\mathcal{N}^{\infty}_{L/K} \circ \mathcal{N}^{\infty}_{E/L})_S \circ \iota^{(S)}_{\lambda} &= \mathcal{N}^{\infty}_{L/K} \circ (q_\mu \circ \gamma_{E_\lambda/L_\mu}) \\ &= (\mathcal{N}^{\infty}_{L/K} \circ q_\mu) \circ \gamma_{E_\lambda/L_\mu} \\ &= (\mathcal{N}^{\infty}_{L/K})_\mu \circ \gamma_{E_\lambda/L_\mu} \\ &= q_\nu \circ \gamma_{L_\mu/K_\nu} \circ \gamma_{E_\lambda/L_\mu},\end{aligned}$$

where $\nu \in \mathbb{a}_K$ is the infinite prime of K defined by $\nu = \lambda \mid_K = \mu \mid_K$. Now, by the transitivity rule

$$\gamma_{L_\mu/K_\nu} \circ \gamma_{E_\lambda/L_\mu} = \gamma_{E_\lambda/K_\nu},$$

it follows that

$$(\mathcal{N}^{\infty}_{E/K})_S \circ \iota^{(S)}_{\lambda} = (\mathcal{N}^{\infty}_{L/K} \circ \mathcal{N}^{\infty}_{E/L})_S \circ \iota^{(S)}_{\lambda},$$

for $\lambda \in \mathbb{a}_E$, which completes the proof. □

Theorem 4.9. *Let L be a finite extension of the global field K. Then, the following square*

$$\begin{array}{ccc}
\mathscr{J}_L^{\varphi_L} & \xrightarrow{\mathsf{NR}_L^{\varphi_L^{\text{Weil}}}} & W_L \\
\Big\downarrow{\scriptstyle \mathscr{N}_{L/K}^{\infty}} & & \Big\downarrow{\scriptstyle \gamma_{L/K}} \\
\mathscr{J}_K^{\varphi_K} & \xrightarrow[\mathsf{NR}_K^{\varphi_K^{\text{Weil}}}]{} & W_K
\end{array}$$

is commutative.

Proof. It suffices to prove that

$$(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} = (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\text{Weil}}})_S \circ \iota_\mu^{(S)},$$

where $\mu \in \mathbb{h}_L \cup \mathbb{a}_L$ and S is any finite subset of $\mathbb{h}_L \cup \mathbb{a}_L$ satisfying $\mu \in S$ and $\mathbb{a}_L \subset S$. So, let μ be any prime of L and S any such subset of $\mathbb{h}_L \cup \mathbb{a}_L$. Then, clearly the following identities

$$\begin{aligned}
(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= (\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty}) \circ q_\mu \\
&= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty} \circ q_\mu) \\
&= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty})_\mu
\end{aligned}$$

hold. Now, first assume that $\mu \in \mathbb{h}_L$. Then,

$$\begin{aligned}
(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty})_\mu \\
&= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (q_\nu \circ \mathscr{N}_{L_\mu/K_\nu}^{\infty}) \\
&= (\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ q_\nu) \circ \mathscr{N}_{L_\mu/K_\nu}^{\infty} \\
&= (e_\nu^{\text{Weil}} \circ \{\bullet, K_\nu\}_{\varphi_{K_\nu}}) \circ \mathscr{N}_{L_\mu/K_\nu}^{\infty} \\
&= e_\nu^{\text{Weil}} \circ (\{\bullet, K_\nu\}_{\varphi_{K_\nu}} \circ \mathscr{N}_{L_\mu/K_\nu}^{\infty}),
\end{aligned}$$

where $\nu \in \mathbb{h}_K$ is the finite prime of K defined by $\nu = \mu \cap O_K$. Now, the diagram (4.10) (equivalently, the square (7.4) of [6]); namely, the following square

$$\begin{array}{ccc}
W_{L_\mu} & \xleftarrow[\sim]{\{\bullet, L_\mu\}_{\varphi_{L_\mu}}} & {}_{\mathbb{Z}}\nabla_{L_\mu}^{(\varphi_{L_\mu})} \\
\Big\downarrow{\scriptstyle \gamma_{L_\mu/K_\nu}} & & \Big\downarrow{\scriptstyle \mathscr{N}_{L_\mu/K_\nu}^{\infty}} \\
W_{K_\nu} & \xleftarrow[\{\bullet, K_\nu\}_{\varphi_{K_\nu}}]{\sim} & {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}
\end{array}$$

is commutative. Therefore,

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= e_\nu^{\text{Weil}} \circ (\{\bullet, K_\nu\}_{\varphi_{K_\nu}} \circ \mathscr{N}_{L_\mu/K_\nu}^{\infty}) \\ &= e_\nu^{\text{Weil}} \circ (\gamma_{L_\mu/K_\nu} \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}}) \\ &= (e_\nu^{\text{Weil}} \circ \gamma_{L_\mu/K_\nu}) \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}}.\end{aligned}$$

Now, by commutative square (4.8) of Lemma 4.6, continuing the computation,

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= (e_\nu^{\text{Weil}} \circ \gamma_{L_\mu/K_\nu}) \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}} \\ &= (\gamma_{L/K} \circ e_\mu^{\text{Weil}}) \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}} \\ &= \gamma_{L/K} \circ (e_\mu^{\text{Weil}} \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}}) \\ &= \gamma_{L/K} \circ (\mathsf{NR}_L^{\varphi_L^{\text{Weil}}} \circ q_\mu),\end{aligned}$$

where the last equality follows from the local-global compatibility of the non-abelian norm residue symbols. Therefore,

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \gamma_{L/K} \circ (\mathsf{NR}_L^{\varphi_L^{\text{Weil}}} \circ q_\mu) \\ &= (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\text{Weil}}}) \circ q_\mu \\ &= (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\text{Weil}}}) \circ c_S \circ \iota_\mu^{(S)} \\ &= (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\text{Weil}}})_S \circ \iota_\mu^{(S)}.\end{aligned}$$

Now, if $\mu \in \mathbb{O}_L$, then

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty})_\mu \\ &= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (q_\nu \circ \gamma_{L_\mu/K_\nu}),\end{aligned}$$

by the equality $(\mathscr{N}_{L/K}^{\infty})_\mu = q_\nu \circ \gamma_{L_\mu/K_\nu}$, where $\nu \in \mathbb{O}_K$ is the infinite prime of K defined by $\nu = \mu\mid_K$. Thus,

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ (q_\nu \circ \gamma_{L_\mu/K_\nu}) \\ &= (\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ q_\nu) \circ \gamma_{L_\mu/K_\nu} \\ &= e_\nu^{\text{Weil}} \circ \gamma_{L_\mu/K_\nu},\end{aligned}$$

as $(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}})_\nu = \mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ q_\nu = e_\nu^{\text{Weil}}$. Therefore,

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\text{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= e_\nu^{\text{Weil}} \circ \gamma_{L_\mu/K_\nu} \\ &= \gamma_{L/K} \circ e_\mu^{\text{Weil}},\end{aligned} \tag{4.11}$$

where the last equality follows from Lemma 4.6. Note that,

$$(\mathsf{NR}_L^{\varphi_L^{\text{Weil}}})_\mu = \mathsf{NR}_L^{\varphi_L^{\text{Weil}}} \circ q_\mu = e_\mu^{\text{Weil}}. \tag{4.12}$$

Thus, substituting (4.12) into (4.11),

$$\begin{aligned}(\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \gamma_{L/K} \circ e_\mu^{\mathrm{Weil}} \\ &= \gamma_{L/K} \circ (\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}} \circ q_\mu) \\ &= (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}}) \circ q_\mu \\ &= (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}}) \circ c_S \circ \iota_\mu^{(S)} \\ &= (\gamma_{L/K} \circ \mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}})_S \circ \iota_\mu^{(S)},\end{aligned}$$

where $\nu \in \mathbb{O}_K$ is the infinite prime of K defined by $\nu = \mu \mid_K$. This completes the proof. □

4.4. Relative non-abelian global reciprocity laws

Recall that, the absolute Weil group W_K of the global field K comes equipped with a continuous homomorphism $\beta_K : W_K \to G_K$ with dense image. Let L be a finite Galois extension of K. Then, $W_L := \beta_K^{-1}(G_L)$ is an open subgroup of W_K and there exists an isomorphism of topological groups

$$W_K/W_L \xrightarrow[\sim]{\beta_{L/K}} G_K/G_L \xrightarrow[\sim]{\mathrm{res}_L^*} \mathrm{Gal}(L/K),$$

where the left arrow is defined by

$$\beta_{L/K} : w \pmod{W_L} \mapsto \beta_K(w) \pmod{G_L}$$

for every $w \in W_K$, and res_L^* is the isomorphism induced from the surjective homomorphism $\mathrm{res}_L : G_K \to \mathrm{Gal}(L/K)$.

The following two lemmas are trivial but useful.

Lemma 4.10. *Let L be a finite Galois extension of the global field K. Then the following square*

$$\begin{array}{ccc} W_K & \xrightarrow{\mathrm{red}_{W_L}} & W_K/W_L \\ {\scriptstyle\beta_K}\downarrow & & \downarrow{\scriptstyle\beta_{L/K}} \\ G_K & \xrightarrow[\mathrm{red}_{G_L}]{} & G_K/G_L \end{array} \tag{4.13}$$

is commutative, where the top and the bottom horizontal arrows are the reduction modulo W_L and the reduction modulo G_L morphisms respectively.

Lemma 4.11. *Let $K \subseteq L \subseteq E$ be a tower of finite Galois extensions of the global field K. Then, the following diagram*

$$\begin{array}{ccc} W_L & \xrightarrow{\mathrm{red}_{W_E}} & W_L/W_E \\ \gamma_{L/K}\downarrow & & \downarrow\beta_{E/L} \\ W_K & & G_L/G_E \\ \mathrm{red}_{W_E}\downarrow & & \downarrow\mathrm{res}_E^* \\ W_K/W_E & & \mathrm{Gal}(E/L) \\ \beta_{E/K}\downarrow & & \downarrow\mathrm{id}_{\mathrm{Gal}(E/L)} \\ G_K/G_E & \xrightarrow[\mathrm{res}_E^*]{} & \mathrm{Gal}(E/K) \end{array} \tag{4.14}$$

is commutative.

Proof. The proof follows from the equality

$$\beta_L(w)\mid_E = \beta_K(w)\mid_E,$$

for every $w \in W_L$. □

For each finite Galois extension L/K, there exists a continuous homomorphism defined by the composition

$$\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \xrightarrow{\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}} W_K \xrightarrow[\text{reduction modulo } W_L]{\mathrm{red}_{W_L}} W_K/W_L \xrightarrow[\sim]{\beta_{L/K}} G_K/G_L \xrightarrow[\sim]{\mathrm{res}_L^*} \mathrm{Gal}(L/K),$$

and called the *non-abelian global reciprocity law relative to the extension L/K*, or the *global non-abelian norm-residue symbol relative to the extension L/K* in this work.

Remark 4.12. This continuous homomorphism is furthermore a surjection if we assume that Conjecture 3.2 holds.

Notation 4.13. Keeping the notation introduced in [4], let $\mathscr{N}_K^{\varphi_K}$ denote the kernel $\ker(\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})$ of the global non-abelian norm-residue symbol $\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to W_K$ of K.

Theorem 4.14. *Assume that Conjecture 3.2 holds. The global non-abelian norm-residue symbol*

$$\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to \mathrm{Gal}(L/K),$$

relative to the finite Galois extension L/K is surjective with open kernel

$$\ker(\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}}) = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}),$$

and induces a topological group isomorphism

$$\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}*}} : \mathscr{I}_K^{\varphi_K}/\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}) \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

Proof. As we have assumed that Conjecture 3.2 holds, the surjectivity of the arrow $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to \mathrm{Gal}(L/K)$ is clear. Moreover, the kernel $\ker(\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}})$ of the continuous surjective homomorphism $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to \mathrm{Gal}(L/K)$ is

$$\ker(\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}}) = (\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})^{-1}(W_L),$$

which proves that $\ker(\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}})$ is open, as W_L is an open subgroup of W_K. Now, the inclusion $\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}) \subseteq (\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})^{-1}(W_L)$ follows from Theorem 4.9. In order to prove the reverse inclusion, let $x \in \mathscr{I}_K^{\varphi_K}$ such that $\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}(x) \in W_L$. As Conjecture 3.2 is assumed to be true, there exists $x' \in \mathscr{I}_L^{\varphi_L}$ such that

$$\gamma_{L/K}\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}}(x') = \mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}(x).$$

Thus, again by Theorem 4.9,

$$\begin{aligned}\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}(x) &= \gamma_{L/K}\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}}(x') \\ &= \mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}(\mathscr{N}_{L/K}^{\infty}(x')).\end{aligned}$$

Therefore there exists $\xi \in \mathscr{N}_K^{\varphi_K}$ such that

$$\xi\mathscr{N}_{L/K}^{\infty}(x') = x.$$

Hence, the inclusion $(\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})^{-1}(W_L) \subseteq \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})$ follows as well. So the equalities

$$\begin{aligned}\ker(\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}}) &= (\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})^{-1}(W_L) \\ &= \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})\end{aligned}$$

hold. Finally, as W_L is an open subgroup of W_K, Conjecture 3.2 implies that the isomorphism

$$\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}^*}} : \mathscr{I}_K^{\varphi_K}/\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}) \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

induced from the non-abelian global reciprocity map

$$\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} : \mathscr{I}_K^{\varphi_K} \to \mathrm{Gal}(L/K)$$

relative to the extension L/K is a topological isomorphism, which completes the proof. □

Remark 4.15. Again assume that Conjecture 3.2 holds. Let L/K be a finite separable (not necessarily Galois) extension. Then, following the same lines of reasoning of the proof of Theorem 4.14, the equality

$$(\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})^{-1}(W_L) = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})$$

follows immediately from Theorem 4.9.

4.5. Relationship with the relative abelian global reciprocity laws

Let L/K be a finite Galois extension. In this subsection, we shall study the "behaviour" of the non-abelian global reciprocity law relative to the extension L/K under the abelianization functor, and prove that under abelianization, the non-abelian global reciprocity law relative to the extension L/K reduces to the abelian global reciprocity law relative to the abelian extension $(L/K)^{ab}/K$.

Theorem 4.16. *The non-abelian global reciprocity law*

$$\mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{L/K} : \mathscr{J}^{\varphi}_{K} \to \mathrm{Gal}(L/K)$$

relative to the extension L/K sits in the following commutative diagram

$$\begin{array}{ccc}
\mathscr{J}^{\varphi}_{K} & \xrightarrow{\mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{L/K}} & \mathrm{Gal}(L/K) \\
{\scriptstyle \mathrm{a}_K}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{(L/K)^{ab}}} \\
\mathbb{J}_K & \xrightarrow[(\bullet, L/K)]{} & \mathrm{Gal}((L/K)^{ab}/K).
\end{array}$$

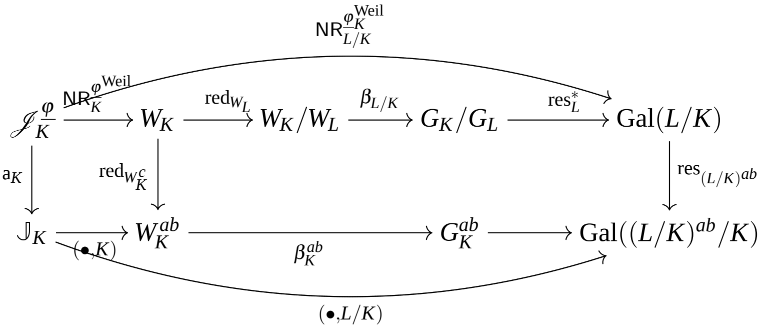
Proof. By Theorem 4.3, the diagram

$$\begin{array}{ccc}
\mathscr{J}^{\varphi}_{K} & \xrightarrow{\mathsf{NR}^{\varphi^{\mathrm{Weil}}}_{K}} & W_K \\
{\scriptstyle \mathrm{a}_K}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_{W_K^c}} \\
\mathbb{J}_K & \xrightarrow[(\bullet, K)]{} & W_K^{ab}
\end{array}$$

is commutative. Moreover, the diagram

$$\begin{array}{ccccccc}
W_K & \xrightarrow{\mathrm{red}_{W_L}} & W_K/W_L & \xrightarrow{\beta_{L/K}} & G_K/G_L & \xrightarrow{\mathrm{res}_L^*} & \mathrm{Gal}(L/K) \\
{\scriptstyle \mathrm{red}_{W_K^c}}\downarrow & & & & & & \downarrow{\scriptstyle \mathrm{res}_{(L/K)^{ab}}} \\
W_K^{ab} & & \xrightarrow[\beta_K^{ab}]{} & & G_K^{ab} & \longrightarrow & \mathrm{Gal}((L/K)^{ab}/K)
\end{array}$$

is trivially commutative, because $\mathrm{res}_{(L/K)^{ab}}(\beta_K(w)\mid_L) = \beta_K(w)\mid_{(L/K)^{ab}}$, for every $w \in W_K$. Thus, the commutativity of the diagram



follows, completing the proof. □

4.6. Decomposition and inertia groups

Let L/K be a finite Galois extension. For $v \in \mathbb{h}_K \cup \mathbb{a}_K$, let $D_v := \mathrm{D}_v(L/K)$ and $I_v := \mathrm{I}_v(L/K)$ denote respectively the decomposition and the inertia groups of v in $\mathrm{Gal}(L/K)$ determined by the continuous homomorphism $e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$. That is, the subgroups D_v and I_v of $\mathrm{Gal}(L/K)$ are defined by

$$D_v = \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}}(W_{K_v})$$

and

$$I_v = \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}}(W_{K_v}^0).$$

Recall that, for $v \in \mathbb{a}_K$, the group $W_{K_v}^0$ is defined by $W_{K_v}^0 = W_{K_v}$.

Theorem 4.17. *For every prime $v \in \mathbb{h}_K \cup \mathbb{a}_K$:*

(i) *The image of the continuous homomorphism defined by the composition*

$$\left(\mathscr{I}_K^{\varphi_K}\right)_v := \left\{\begin{array}{ll} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}, & v \in \mathbb{h}_K \\ W_{\mathbb{R}}, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}, & v \in \mathbb{a}_{K,\mathbb{C}} \end{array}\right\} \xrightarrow{q_v} \mathscr{I}_K^{\varphi_K} \xrightarrow{\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}}} \mathrm{Gal}(L/K)$$

is the decomposition group D_v of v in $\mathrm{Gal}(L/K)$ *determined by the continuous homomorphism $e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$.*

(ii) *The image of the continuous homomorphism defined by the composition*

$$\left(\mathscr{I}_K^{\varphi_K}\right)_v^0 := \left\{\begin{array}{ll} {}_{1}\nabla_{K_v}^{(\varphi_{K_v})^0}, & v \in \mathbb{h}_K \\ W_{\mathbb{R}}^0, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}^0, & v \in \mathbb{a}_{K,\mathbb{C}} \end{array}\right\} \xrightarrow{q_v} \mathscr{I}_K^{\varphi_K} \xrightarrow{\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}}} \mathrm{Gal}(L/K)$$

is the inertia group I_v of v in $\mathrm{Gal}(L/K)$ *determined by the continuous homomorphism $e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$.*

Proof. First, assume that $v \in \mathbb{h}_K$. Then, clearly

$$\begin{aligned}\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v &= (\mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ \mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}) \circ q_v \\ &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ (\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} \circ q_v) \\ &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ (e_v^{\mathrm{Weil}} \circ \{\bullet, K_v\}_{\varphi_{K_v}})\end{aligned}$$

by the local-global compatibility of the non-abelian norm-residue symbols discussed in Subsection 4.1. Thus, the image $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})})$ of $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v$ is given by

$$\begin{aligned}\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}) &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}} \circ \{\bullet, K_v\}_{\varphi_{K_v}}({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}) \\ &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}}(W_{K_v}) \\ &= D_v\end{aligned}$$

which is the decomposition group of v in $\mathrm{Gal}(L/K)$ determined by $e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$, and the image $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v({}_1\nabla_{K_v}^{(\varphi_{K_v})^0})$ of $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v$ is given by

$$\begin{aligned}\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v({}_1\nabla_{K_v}^{(\varphi_{K_v})^0}) &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}} \circ \{\bullet, K_v\}_{\varphi_{K_v}}({}_1\nabla_{K_v}^{(\varphi_{K_v})^0}) \\ &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}}(W_{K_v}^0) \\ &= I_v\end{aligned}$$

which is the inertia group of v in $\mathrm{Gal}(L/K)$ determined by $e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$. Next, assume that $v \in \mathbb{a}_K$. Then,

$$\begin{aligned}\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v &= (\mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ \mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}}) \circ q_v \\ &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ (\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} \circ q_v) \\ &= \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}},\end{aligned}$$

and the image of $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v$ is given by

$$\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v(W_{K_v}) = \mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L} \circ e_v^{\mathrm{Weil}}(W_{K_v}) = D_v = I_v,$$

which completes the proof. □

Remark 4.18. Alternatively, we can prove the archimedean case of Theorem 4.17 as follows.

If $v \in \mathbb{a}_{K,\mathbb{C}}$; that is, v is a complex archimedean prime of K, then the decomposition and the inertia groups D_v and I_v of v in $\mathrm{Gal}(L/K)$ determined by the continuous homomorphism $e_v^{\mathrm{Weil}} : W_{\mathbb{C}} := \mathbb{C}^\times \to W_K$ are the same. Moreover, $D_v = I_v$ is the trivial subgroup $\langle \mathrm{id}_L \rangle$ of $\mathrm{Gal}(L/K)$, and the image of $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v$ is given by

$$\begin{aligned}\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v(W_{\mathbb{C}}) &= \left\{ \beta_K(e_v^{\mathrm{Weil}}(w)) \mid_L : w \in W_{\mathbb{C}} \right\} \\ &= \left\{ e_v^{\mathrm{Galois}}(\beta_{\mathbb{C}}(w)) \mid_L : w \in W_{\mathbb{C}} \right\} = \langle \mathrm{id}_L \rangle = D_v = I_v\end{aligned}$$

where the equalities in the last line follow from the commutativity of the square (4.5) given by

$$\begin{array}{ccc} W_{\mathbb{C}} & \xrightarrow{e_v^{\mathrm{Weil}}} & W_K \\ {\scriptstyle \beta_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \beta_K} \\ G_{\mathbb{C}} & \xrightarrow[e_v^{\mathrm{Galois}}]{} & G_K \end{array}$$

and from the fact that $\beta_{\mathbb{C}} : W_{\mathbb{C}} \to G_{\mathbb{C}}$ is the trivial mapping.

If $v \in \infty_{K,\mathbb{R}}$; that is, v is a real archimedean prime of K, then the decomposition and the inertia groups D_v and I_v of v in $\mathrm{Gal}(L/K)$ determined by the continuous homomorphism $e_v^{\mathrm{Weil}} : W_{\mathbb{R}} := \mathbb{C}^\times \cup j\mathbb{C}^\times \to W_K$ are the same, and $D_v = I_v$ is the subgroup $\left\langle e_v^{\mathrm{Galois}}(c)\mid_L \right\rangle$ of $\mathrm{Gal}(L/K)$, which is of order at most 2. Here, $c : \mathbb{C} \to \mathbb{C}$ denotes the $\mathbb{R}$-automorphism of $\mathbb{C}$ defined by the complex conjugation $c : z \mapsto \bar{z}$, for $z \in \mathbb{C}$. (For example, if K is a totally real number field and L/K is a totally real extension, then $e_v^{\mathrm{Galois}}(c)\mid_L = \mathrm{id}_L$.) Therefore, the image of $\mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v$ is

$$\begin{aligned} \mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_v(W_{\mathbb{R}}) &= \left\{ \beta_K(e_v^{\mathrm{Weil}}(w))\mid_L : w \in W_{\mathbb{R}} \right\} \\ &= \left\{ e_v^{\mathrm{Galois}}(\beta_{\mathbb{R}}(w))\mid_L : w \in W_{\mathbb{R}} \right\} \\ &= \left\{ \mathrm{id}_L, e_v^{\mathrm{Galois}}(c)\mid_L \right\} = D_v = I_v, \end{aligned}$$

as $\beta_{\mathbb{R}}(\mathbb{C}^\times) = \mathrm{id}_{\mathbb{C}}$ and $\beta_{\mathbb{R}}(j\mathbb{C}^\times) = c$ (look at 1.4.3 of [14] for details), which completes the proof.

4.7. Basic functorial properties

Theorem 4.19. *Let $K \subseteq L \subseteq E$ be a tower of finite Galois extensions of the global field K. Then, the triangle*

$$\begin{array}{ccc} \mathscr{I}_K^{\varphi_K} & \xrightarrow{\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}}} & \mathrm{Gal}(E/K) \\ & {\scriptstyle \mathsf{NR}_{L/K}^{\varphi_K^{\mathrm{Weil}}}}\searrow & \downarrow{\scriptstyle \mathrm{res}_L} \\ & & \mathrm{Gal}(L/K) \end{array}$$

where the right vertical arrow is the restriction map to L, is commutative.

Proof. First note that, for the tower $K \subseteq L \subseteq E$ of finite Galois extensions of K, the following rectangle

$$
\begin{array}{ccccc}
W_K/W_E & \xrightarrow[\sim]{\beta_{E/K}} & G_K/G_E & \xrightarrow[\sim]{\mathrm{res}_E^*} & \mathrm{Gal}(E/K) \\
\downarrow & & \downarrow & & \downarrow \mathrm{res}_L \\
W_K/W_L & \xrightarrow[\beta_{L/K}]{\sim} & G_K/G_L & \xrightarrow[\mathrm{res}_L^*]{\sim} & \mathrm{Gal}(L/K)
\end{array}
$$

is commutative. Thus, it suffices to prove the commutativity of the triangle

$$
\begin{array}{ccc}
\mathscr{I}_K^{\varphi_K} & \xrightarrow{\mathrm{red}_{W_E} \circ \mathrm{NR}_K^{\varphi_K^{\mathrm{Weil}}}} & W_K/W_E \\
& \searrow^{\mathrm{red}_{W_L} \circ \mathrm{NR}_K^{\varphi_K^{\mathrm{Weil}}}} & \downarrow \\
& & W_K/W_L
\end{array}
$$

which is clear. $\square$

Theorem 4.20. *Let $K \subseteq L \subseteq E$ be a tower of finite Galois extensions of the global field K. Then, the square*

$$
\begin{array}{ccc}
\mathscr{I}_L^{\varphi_L} & \xrightarrow{\mathrm{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}}} & \mathrm{Gal}(E/L) \\
\mathscr{N}_{L/K}^{\infty} \downarrow & & \downarrow \mathrm{id}_{\mathrm{Gal}(E/L)} \\
\mathscr{I}_K^{\varphi_K} & \xrightarrow[\mathrm{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}}]{} & \mathrm{Gal}(E/K)
\end{array}
$$

is commutative.

Proof. It suffices to prove that

$$(\mathrm{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} = (\mathrm{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}})_S \circ \iota_\mu^{(S)},$$

where $\mu \in \mathbb{h}_L \cup \mathbb{a}_L$ and S is any finite subset of $\mathbb{h}_L \cup \mathbb{a}_L$ satisfying $\mu \in S$ and $\mathbb{a}_L \subset S$. So, let μ be any prime of L and S any such subset of $\mathbb{h}_L \cup \mathbb{a}_L$. Then, clearly the following identities

$$
\begin{aligned}
(\mathrm{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= (\mathrm{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty}) \circ q_\mu \\
&= \mathrm{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty} \circ q_\mu) \\
&= \mathrm{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty})_\mu
\end{aligned}
$$

hold. Now, first assume that $\mu \in \mathbb{h}_L$. Then,

$$\begin{aligned}
(\mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{E/K} \circ \mathcal{N}^{\infty}_{L/K})_S \circ \iota^{(S)}_{\mu} &= \mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{E/K} \circ (\mathcal{N}^{\infty}_{L/K})_{\mu} \\
&= \mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{E/K} \circ (q_{\nu} \circ \mathcal{N}^{\infty}_{L_\mu/K_\nu}) \\
&= (\mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{E/K} \circ q_{\nu}) \circ \mathcal{N}^{\infty}_{L_\mu/K_\nu} \\
&= (\mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ \mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{K} \circ q_{\nu}) \circ \mathcal{N}^{\infty}_{L_\mu/K_\nu} \\
&= (\mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ e^{\mathrm{Weil}}_{\nu} \circ \{\bullet, K_\nu\}_{\varphi_{K_\nu}}) \circ \mathcal{N}^{\infty}_{L_\mu/K_\nu} \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ e^{\mathrm{Weil}}_{\nu} \circ (\{\bullet, K_\nu\}_{\varphi_{K_\nu}} \circ \mathcal{N}^{\infty}_{L_\mu/K_\nu}),
\end{aligned}$$

where $\nu \in \mathbb{h}_K$ is the finite prime of K defined by $\nu = \mu \cap O_K$. Now, the diagram (4.10) (equivalently, the square (7.4) of [6]); that is, the following square

$$\begin{array}{ccc}
W_{L_\mu} & \xleftarrow[\sim]{\{\bullet, L_\mu\}_{\varphi_{L_\mu}}} & {}_{\mathbb{Z}}\nabla^{(\varphi_{L_\mu})}_{L_\mu} \\
\Big\downarrow{\scriptstyle \gamma_{L_\mu/K_\nu}} & & \Big\downarrow{\scriptstyle \mathcal{N}^{\infty}_{L_\mu/K_\nu}} \\
W_{K_\nu} & \xleftarrow[\{\bullet, K_\nu\}_{\varphi_{K_\nu}}]{\sim} & {}_{\mathbb{Z}}\nabla^{(\varphi_{K_\nu})}_{K_\nu}
\end{array}$$

is commutative. Therefore,

$$\begin{aligned}
(\mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{E/K} \circ \mathcal{N}^{\infty}_{L/K})_S \circ \iota^{(S)}_{\mu} &= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ e^{\mathrm{Weil}}_{\nu} \circ (\{\bullet, K_\nu\}_{\varphi_{K_\nu}} \circ \mathcal{N}^{\infty}_{L_\lambda/K_\nu}) \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ e^{\mathrm{Weil}}_{\nu} \circ (\gamma_{L_\mu/K_\nu} \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}}) \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ (e^{\mathrm{Weil}}_{\nu} \circ \gamma_{L_\mu/K_\nu}) \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}}.
\end{aligned}$$

Now, by commutative square (4.8) of Lemma 4.6, continuing the computation,

$$\begin{aligned}
(\mathsf{NR}^{\varphi_K^{\mathrm{Weil}}}_{E/K} \circ \mathcal{N}^{\infty}_{L/K})_S \circ \iota^{(S)}_{\mu} &= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ (e^{\mathrm{Weil}}_{\nu} \circ \gamma_{L_\mu/K_\nu}) \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}} \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ (\gamma_{L/K} \circ e^{\mathrm{Weil}}_{\mu}) \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}} \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ \gamma_{L/K} \circ (e^{\mathrm{Weil}}_{\mu} \circ \{\bullet, L_\mu\}_{\varphi_{L_\mu}}) \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ \gamma_{L/K} \circ (\mathsf{NR}^{\varphi_L^{\mathrm{Weil}}}_{L} \circ q_{\mu}) \\
&= \mathrm{res}^*_E \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ \gamma_{L/K} \circ (\mathsf{NR}^{\varphi_L^{\mathrm{Weil}}}_{L} \circ c_S \circ \iota^{(S)}_{\mu}) \\
&= \mathrm{res}^*_E \circ \beta_{E/L} \circ \mathrm{red}_{W_E} \circ (\mathsf{NR}^{\varphi_L^{\mathrm{Weil}}}_{L} \circ c_S \circ \iota^{(S)}_{\mu}),
\end{aligned}$$

where the last equality follows from Lemma 4.11. Thus,

$$\begin{aligned}(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathrm{res}_E^* \circ \beta_{E/L} \circ \mathrm{red}_{W_E} \circ (\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}} \circ c_S \circ \iota_\mu^{(S)}) \\ &= (\mathrm{res}_E^* \circ \beta_{E/L} \circ \mathrm{red}_{W_E} \circ \mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}}) \circ c_S \circ \iota_\mu^{(S)} \\ &= \mathsf{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}} \circ c_S \circ \iota_\mu^{(S)} \\ &= (\mathsf{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}})_S \circ \iota_\mu^{(S)}.\end{aligned}$$

Now, if $\mu \in \mathbb{O}_L$, then

$$\begin{aligned}(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ (\mathscr{N}_{L/K}^{\infty})_\mu \\ &= \mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ (q_\nu \circ \gamma_{L_\mu/K_\nu}),\end{aligned}$$

by the equality $(\mathscr{N}_{L/K}^{\infty})_\mu = q_\nu \circ \gamma_{L_\mu/K_\nu}$, where $\nu \in \mathbb{O}_K$ is the infinite prime of K defined by $\nu = \mu\mid_K$. Then, by the definition of the non-abelian global reciprocity law $\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}}$ relative to the extension E/K,

$$\begin{aligned}(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ (q_\nu \circ \gamma_{L_\mu/K_\nu}) \\ &= (\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ q_\nu) \circ \gamma_{L_\mu/K_\nu} \\ &= (\mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ \mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} \circ q_\nu) \circ \gamma_{L_\mu/K_\nu} \\ &= (\mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ e_\nu^{\mathrm{Weil}}) \circ \gamma_{L_\mu/K_\nu},\end{aligned}$$

as $(\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})_\nu = \mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}} \circ q_\nu = e_\nu^{\mathrm{Weil}}$. Therefore,

$$\begin{aligned}(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= (\mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ e_\nu^{\mathrm{Weil}}) \circ \gamma_{L_\mu/K_\nu} \\ &= \mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ (e_\nu^{\mathrm{Weil}} \circ \gamma_{L_\mu/K_\nu}) \\ &= \mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ (\gamma_{L/K} \circ e_\mu^{\mathrm{Weil}}),\end{aligned} \tag{4.15}$$

where the last equality follows from Lemma 4.6. Note that,

$$(\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}})_\mu = \mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}} \circ q_\mu = e_\mu^{\mathrm{Weil}}. \tag{4.16}$$

Thus, substituting (4.16) into (4.15),

$$\begin{aligned}(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathscr{N}_{L/K}^{\infty})_S \circ \iota_\mu^{(S)} &= \mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ (\gamma_{L/K} \circ e_\mu^{\mathrm{Weil}}) \\ &= \mathrm{res}_E^* \circ \beta_{E/K} \circ \mathrm{red}_{W_E} \circ \gamma_{L/K} \circ (\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}} \circ q_\mu) \\ &= \mathrm{res}_E^* \circ \beta_{E/L} \circ \mathrm{red}_{W_E} \circ (\mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}} \circ q_\mu),\end{aligned}$$

where the last equality follows from Lemma 4.11. Thus, using the definition of the non-abelian global reciprocity law $\mathsf{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}}$ relative to the extension E/L,

$$\begin{aligned}(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Weil}}} \circ \mathcal{N}_{L/K}^{\infty})_S \circ \iota_{\mu}^{(S)} &= (\mathrm{res}_E^* \circ \beta_{E/L} \circ \mathrm{red}_{W_E} \circ \mathsf{NR}_L^{\varphi_L^{\mathrm{Weil}}}) \circ q_{\mu} \\ &= \mathsf{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}} \circ q_{\mu} \\ &= \mathsf{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}} \circ c_S \circ \iota_{\mu}^{(S)} \\ &= (\mathsf{NR}_{E/L}^{\varphi_L^{\mathrm{Weil}}})_S \circ \iota_{\mu}^{(S)},\end{aligned}$$

where $\nu \in \infty_K$ is the infinite prime of K defined by $\nu = \mu\mid_K$. This completes the proof. □

4.8. Non-abelian global existence theorem

In this subsection, under the assumption that Conjecture 3.2 holds, we shall prove the following theorem, which is the non-abelian generalization of the existence theorem of abelian global class field theory. As usual, K denotes a global field.

Theorem 4.21 (Non-abelian global existence theorem). *There exists an inclusion-reversing bijective correspondence*

$$\left\{\begin{matrix}\text{Finite Galois exten–}\\ \text{sions of } K \text{ inside } K^{sep}\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{Open normal subgroups of finite}\\ \text{index in } \mathcal{J}_K^{\varphi_K} \text{ containing } \mathcal{N}_K^{\varphi_K}\end{matrix}\right\}$$

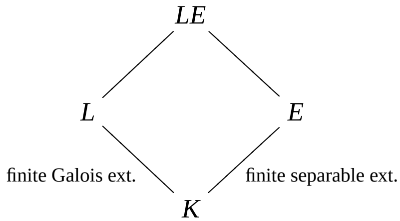
defined by

$$L \mapsto \mathcal{N}_K^{\varphi_K}\mathcal{N}_{L/K}^{\infty}(\mathcal{J}_L^{\varphi_L}),$$

for each finite Galois extension L of K inside K^{sep}, where $\mathcal{N}_K^{\varphi_K} = \ker(\mathsf{NR}_K^{\varphi_K^{\mathrm{Weil}}})$.

In order to prove the injectivity part of the non-abelian global existence theorem, we shall first prove the following lemma :

Lemma 4.22. *Let L/K be a finite Galois extension and E/K any finite separable extension.*



Then,

$$\mathcal{N}_E^{\varphi_E}\mathcal{N}_{LE/E}^{\infty}(\mathcal{J}_{LE}^{\varphi_{LE}}) = (\mathcal{N}_{E/K}^{\infty})^{-1}\left(\mathcal{N}_K^{\varphi_K}\mathcal{N}_{L/K}^{\infty}(\mathcal{J}_L^{\varphi_L})\right),$$

and the rectangle

$$\begin{array}{ccc}
\mathscr{I}_E^{\varphi_E}/\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}}) & \xrightarrow[\sim]{{\mathsf{NR}_{LE/E}^{\varphi_E}}^{\mathrm{Weil}*}} & \mathrm{Gal}(LE/E) \\
\Big\downarrow{\scriptstyle (\mathrm{red}_{\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})}\circ\mathscr{N}_{E/K}^{\infty})^*} & & \Big\downarrow{\scriptstyle \mathrm{res}_L} \\
\mathscr{I}_K^{\varphi_K}/\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}) & \xrightarrow[{\mathsf{NR}_{L/K}^{\varphi_K}}^{\mathrm{Weil}*}]{\sim} & \mathrm{Gal}(L/K)
\end{array}$$

is commutative.

Proof. In fact, the following diagram

$$\begin{array}{ccccccc}
W_E & \xrightarrow{\mathrm{red}_{W_{LE}}} & W_E/W_{LE} & \xrightarrow[\sim]{\beta_{LE/E}} & G_E/G_{LE} & \xrightarrow[\sim]{\mathrm{res}_{LE}^*} & \mathrm{Gal}(LE/E) \\
\Big\downarrow{\scriptstyle \gamma_{E/K}} & & & & & & \Big\downarrow{\scriptstyle \mathrm{res}_L} \\
W_K & \xrightarrow{\mathrm{red}_{W_L}} & W_K/W_L & \xrightarrow[\sim]{\beta_{L/K}} & G_K/G_L & \xrightarrow[\sim]{\mathrm{res}_L^*} & \mathrm{Gal}(L/K)
\end{array}$$

is commutative, because for each $w \in W_E$, the equalities

$$\mathrm{res}_L(\mathrm{res}_{LE}(\beta_E(w))) = \mathrm{res}_L(\beta_E(w)) = \mathrm{res}_L(\beta_K \circ \gamma_{E/K}(w))$$

hold by (4.2). Thus, by Theorem 4.9, the diagram

$$\begin{array}{ccccccccc}
 & & & & \overset{\mathsf{NR}_{LE/E}^{\varphi_E\,\mathrm{Weil}}}{\frown} & & & & \\
\mathscr{I}_E^{\varphi_E} & \xrightarrow{\mathsf{NR}_E^{\varphi_E\,\mathrm{Weil}}} & W_E & \xrightarrow{\mathrm{red}_{W_{LE}}} & W_E/W_{LE} & \xrightarrow[\sim]{\beta_{LE/E}} & G_E/G_{LE} & \xrightarrow[\sim]{\mathrm{res}_{LE}^*} & \mathrm{Gal}(LE/E) \\
\Big\downarrow{\scriptstyle \mathscr{N}_{E/K}^{\infty}} & & \Big\downarrow{\scriptstyle \gamma_{E/K}} & & & & & & \Big\downarrow{\scriptstyle \mathrm{res}_L} \\
\mathscr{I}_K^{\varphi_K} & \xrightarrow{\mathsf{NR}_K^{\varphi_K\,\mathrm{Weil}}} & W_K & \xrightarrow{\mathrm{red}_{W_L}} & W_K/W_L & \xrightarrow[\sim]{\beta_{L/K}} & G_K/G_L & \xrightarrow[\sim]{\mathrm{res}_L^*} & \mathrm{Gal}(L/K) \\
 & & & & \underset{\mathsf{NR}_{L/K}^{\varphi_K\,\mathrm{Weil}}}{\smile} & & & &
\end{array} \tag{4.17}$$

is commutative. Now, if the equality

$$\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}}) = (\mathscr{N}_{E/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})\right)$$

holds, then the composite homomorphism

$$\mathscr{I}_E^{\varphi_E} \xrightarrow{\mathscr{N}_{E/K}^{\infty}} \mathscr{I}_K^{\varphi_K} \xrightarrow{\mathrm{red}_{\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})}} \mathscr{I}_K^{\varphi_K}/\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})$$

naturally induces the well-defined group homomorphism

$$\mathscr{I}_E^{\varphi_E}/\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}}) \xrightarrow{(\mathrm{red}_{\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})}\circ\mathscr{N}_{E/K}^{\infty})^*} \mathscr{I}_K^{\varphi_K}/\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})$$

as $\ker\left(\mathrm{red}_{\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})}\circ\mathscr{N}_{E/K}^{\infty}\right)=\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}})$, and the commutative rectangle (4.17) yields the commutativity of the diagram

$$\begin{array}{ccc}
\mathscr{I}_E^{\varphi_E}/\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}}) & \xrightarrow[\sim]{\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}^*}} & \mathrm{Gal}(LE/E) \\
\Big\downarrow{\scriptstyle (\mathrm{red}_{\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})}\circ\mathscr{N}_{E/K}^{\infty})^*} & & \Big\downarrow{\scriptstyle \mathrm{res}_L} \\
\mathscr{I}_K^{\varphi_K}/\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}) & \xrightarrow[\mathsf{NR}_{L/K}^{\varphi_K\,\,\mathrm{Weil}^*}]{\sim} & \mathrm{Gal}(L/K).
\end{array}$$

Thus, it remains to prove that the equality

$$\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}})=(\mathscr{N}_{E/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})\right)$$

holds. In order to do so, let

$$\alpha\in\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}})=\ker(\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}),$$

where the equality follows from Theorem 4.14. Then

$$\mathsf{NR}_{L/K}^{\varphi_K\,\,\mathrm{Weil}}(\mathscr{N}_{E/K}^{\infty}(\alpha))=\mathrm{res}_L\circ\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}(\alpha)=\mathrm{id}_L$$

by the commutativity of the diagram (4.17) and by the choice of α. Therefore, it follows that

$$\alpha\in(\mathscr{N}_{E/K}^{\infty})^{-1}(\ker(\mathsf{NR}_{L/K}^{\varphi_K\,\,\mathrm{Weil}}))=(\mathscr{N}_{E/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})\right),$$

where the equality follows from Theorem 4.14. For the reverse inclusion, assume that

$$\alpha\in(\mathscr{N}_{E/K}^{\infty})^{-1}(\ker(\mathsf{NR}_{L/K}^{\varphi_K\,\,\mathrm{Weil}}))=(\mathscr{N}_{E/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})\right),$$

where the equality follows from Theorem 4.14. Therefore,

$$\mathsf{NR}_{L/K}^{\varphi_K\,\,\mathrm{Weil}}(\mathscr{N}_{E/K}^{\infty}(\alpha))=\mathrm{res}_L\circ\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}(\alpha)=\mathrm{id}_L$$

by the commutativity of the diagram (4.17) and by the choice of α. Thus, $\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}(\alpha)\in\mathrm{Gal}(LE/E)$ such that

$$\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}(\alpha)\mid_L=\mathrm{id}_L \text{ and } \mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}(\alpha)\mid_E=\mathrm{id}_E,$$

proving that $\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}}(\alpha)=\mathrm{id}_{LE}$. So, it follows that,

$$\alpha\in\ker(\mathsf{NR}_{LE/E}^{\varphi_E\,\,\mathrm{Weil}})=\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{I}_{LE}^{\varphi_{LE}}),$$

where the equality follows from Theorem 4.14, which completes the proof. □

Now, proof of the injectivity part of the non-abelian global existence theorem follows. In fact, let L and E be any two finite Galois extensions of K inside K^{sep} such that

$$\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{J}_L^{\varphi_L}) = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{E/K}^{\infty}(\mathscr{J}_E^{\varphi_E}).$$

Then, by the previous lemma,

$$\begin{aligned}\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{J}_{LE}^{\varphi_{LE}}) &= (\mathscr{N}_{E/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{J}_L^{\varphi_L})\right)\\ &= (\mathscr{N}_{E/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{E/K}^{\infty}(\mathscr{J}_E^{\varphi_E})\right)\\ &= \mathscr{J}_E^{\varphi_E},\end{aligned}$$

and the isomorphism

$$\mathsf{NR}_{LE/E}^{\varphi_E\ \mathrm{Weil}^*} : \mathscr{J}_E^{\varphi_E}/\mathscr{N}_E^{\varphi_E}\mathscr{N}_{LE/E}^{\infty}(\mathscr{J}_{LE}^{\varphi_{LE}}) \xrightarrow{\sim} \mathrm{Gal}(LE/E)$$

yields the equality $\mathrm{Gal}(LE/E) = \{\mathrm{id}_{LE}\}$. That is, $L \subseteq E$. Replacing the roles of L and E gives the reverse inclusion $E \subseteq L$, which completes the proof.

Next, in order to prove the surjectivity part of the non-abelian global existence theorem, we shall prove the following lemma :

Lemma 4.23. *If S is an open normal subgroup of finite index in $\mathscr{J}_K^{\varphi_K}$ containing $\mathscr{N}_K^{\varphi_K}$, then there exists a finite Galois extension L_o of K inside K^{sep} such that*

$$S = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L_o/K}^{\infty}(\mathscr{J}_{L_o}^{\varphi_{L_o}}).$$

Proof. Let S be an open normal subgroup of finite index in $\mathscr{J}_K^{\varphi_K}$ containing $\mathscr{N}_K^{\varphi_K}$. Denote the open subgroup of finite index in $\mathscr{N}_K^{\varphi_K}\backslash\mathscr{J}_K^{\varphi_K}$ corresponding to S by $\overline{S}$. As Conjecture 3.2 is assumed to be true, the topological isomorphism

$$\mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*} : \mathscr{N}_K^{\varphi_K}\backslash\mathscr{J}_K^{\varphi_K} \xrightarrow{\sim} W_K$$

defines an open normal subgroup $\mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*}(\overline{S})$ of finite index in W_K. Now, choose a finite Galois extension L of K inside K^{sep} such that $W_L \subseteq \mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*}(\overline{S})$. Thus, W_L is a normal subgroup of W_K and $\mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*}(\overline{S})/W_L$ is a normal subgroup of the finite quotient group $W_K/W_L \xrightarrow{\sim} \mathrm{Gal}(L/K)$. Hence under the isomorphism $W_K/W_L \underset{\sim}{\xrightarrow{\beta_{L/K}}} G_K/G_L \underset{\sim}{\xrightarrow{\mathrm{res}_L^*}} \mathrm{Gal}(L/K)$, the normal subgroup $\mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*}(\overline{S})/W_L$ of W_K/W_L is mapped isomorphically onto a normal subgroup $\mathrm{Gal}(L/L_o)$ of $\mathrm{Gal}(L/K)$ for some finite Galois subextension L_o/K of L/K. Now, we claim that

$$\mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*}(\overline{S}) = W_{L_o} = \beta_K^{-1}(G_{L_o}).$$

In order to prove this claim, let $w \in \mathsf{NR}_K^{\varphi_K\ \mathrm{Weil}^*}(\overline{S})$. Then,

$$\mathrm{res}_L^* \circ \beta_{L/K} \circ \mathrm{red}_{W_L}(w) = \mathrm{res}_L^*(\beta_K(w) \pmod{G_L}) = \beta_K(w)\,|_L \in \mathrm{Gal}(L/L_o),$$

which shows that $\beta_K(w)\mid_{L_o} = \mathrm{id}_{L_o}$. Thus, $\beta_K(w) \in G_{L_o}$ and $w \in \beta_K^{-1}(G_{L_o}) = W_{L_o}$, proving that ${\mathsf{NR}_K^{\varphi_K}}^{\mathrm{Weil}^*}(\overline{S})$ is a subgroup of W_{L_o}. Now, it follows that,

$$[L : L_o] = ({\mathsf{NR}_K^{\varphi_K}}^{\mathrm{Weil}^*}(\overline{S}) : W_L) \leq (W_{L_o} : W_L) = [L : L_o],$$

which proves the equality

$${\mathsf{NR}_K^{\varphi_K}}^{\mathrm{Weil}^*}(\overline{S}) = W_{L_o}.$$

Now, following the proof of Theorem 4.14,

$$S = ({\mathsf{NR}_K^{\varphi_K}}^{\mathrm{Weil}})^{-1}(W_{L_o}) = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L_o/K}^{\infty}(\mathscr{I}_{L_o}^{\varphi_{L_o}}),$$

which completes the proof. □

Now, the bijectivity part of Theorem 4.21 follows directly from Lemmas 4.22 and 4.23. Moreover, this bijective correspondence is inclusion-reversing. In fact, let L and E be any two finite Galois extensions of K inside K^{sep}. If $L \subseteq E$, then by Proposition 4.8,

$$\mathscr{N}_{E/K}^{\infty}(\mathscr{I}_E^{\varphi_E}) = \mathscr{N}_{L/K}^{\infty} \circ \mathscr{N}_{E/L}^{\infty}(\mathscr{I}_E^{\varphi_E}) \subseteq \mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}).$$

Therefore,

$$\mathscr{N}_K^{\varphi_K}\mathscr{N}_{E/K}^{\infty}(\mathscr{I}_E^{\varphi_E}) \subseteq \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}),$$

which proves that the bijective correspondence is inclusion-reversing. Hence the proof of Theorem 4.21 is complete now.

Corollary 4.24. *If L is a finite separable extension of K inside K^{sep}, then*

$$S = ({\mathsf{NR}_K^{\varphi_K}}^{\mathrm{Weil}})^{-1}(W_L)$$

is an open subgroup of $\mathscr{I}_K^{\varphi_K}$ containing $\mathscr{N}_K^{\varphi_K}$. Moreover,

$$S = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L_o/K}^{\infty}(\mathscr{I}_{L_o}^{\varphi_{L_o}})$$

where L_o/K is the maximal Galois subextension of L/K.

Proof. Clearly S is an open subgroup of $\mathscr{I}_K^{\varphi_K}$ containing $\mathscr{N}_K^{\varphi_K}$, as W_L is an open subgroup of W_K. By Lemma 4.23, there exists a finite Galois extension L_o of K inside K^{sep} such that

$$S = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L_o/K}^{\infty}(\mathscr{I}_{L_o}^{\varphi_{L_o}}).$$

Now, we claim that L_o/K is the maximal Galois subextension of L/K. In fact, by Lemma 4.22 and Remark 4.15,

$$\begin{aligned}
\mathscr{N}_L^{\varphi_L}\mathscr{N}_{L_oL/L}^{\infty}(\mathscr{I}_{L_oL}^{\varphi_{L_oL}}) &= (\mathscr{N}_{L/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L_o/K}^{\infty}(\mathscr{I}_{L_o}^{\varphi_{L_o}})\right) \\
&= (\mathscr{N}_{L/K}^{\infty})^{-1}(S) \\
&= (\mathscr{N}_{L/K}^{\infty})^{-1}\left(\mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L})\right) \\
&= \mathscr{I}_L^{\varphi_L}.
\end{aligned}$$

Therefore, the isomorphism

$$\mathsf{NR}_{L_oL/L}^{\varphi_L\ \mathrm{Weil}^*} : \mathscr{I}_L^{\varphi_L}/\mathscr{N}_L^{\varphi_L}\mathscr{N}_{L_oL/L}^{\infty}(\mathscr{I}_{L_oL}^{\varphi_{L_oL}}) \xrightarrow{\sim} \mathrm{Gal}(L_oL/L)$$

yields the equality $\mathrm{Gal}(L_oL/L) = \{\mathrm{id}_{L_oL}\}$. That is, $L_o \subseteq L$. Now, let E/K be any Galois subextension of L/K. Then, as $E \subseteq L$,

$$\mathscr{N}_K^{\varphi_K}\mathscr{N}_{E/K}^{\infty}(\mathscr{I}_E^{\varphi_E}) \supseteq \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{I}_L^{\varphi_L}) = S$$

by Proposition 4.8 . Therefore,

$$\mathscr{N}_K^{\varphi_K}\mathscr{N}_{E/K}^{\infty}(\mathscr{I}_E^{\varphi_E}) \supseteq S = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L_o/K}^{\infty}(\mathscr{I}_{L_o}^{\varphi_{L_o}}),$$

which proves that $E \subseteq L_o$ by the existence theorem, completing the proof. □

4.9. Non-abelian ray class groups and non-abelian ray class fields

Let $\mathfrak{m}$ be a cycle (=modulus) of the global field K. So, $\mathfrak{m}$ is a formal product of the form

$$\mathfrak{m} = \mathfrak{m}_{\mathbb{h}_K}\mathfrak{m}_{\mathbb{a}_K}.$$

Here, the "henselian part" $\mathfrak{m}_{\mathbb{h}_K}$ is just a non-zero integral ideal in the ring of integers O_K of K viewed as a product $\prod_{v\in\mathbb{h}_K} v^{e_v}$ of henselian primes v in K. The "archimedean part" $\mathfrak{m}_{\mathbb{a}_K}$ is any subset of the real archimedean primes $\mathbb{a}_{K,\mathbb{R}}$ of K, if K is a number field. On the other hand, as $\mathbb{a}_K = \varnothing$ in case K is a function field, we just define $\mathfrak{m}_{\mathbb{a}_K} = \varnothing$ whenever K is a function field. In both cases we shall identify $\mathfrak{m}_{\mathbb{a}_K}$ with a certain formal product $\prod_{v\in\mathbb{a}_K} v^{e_v}$ (so that $\mathfrak{m} = \mathfrak{m}_{\mathbb{h}_K}$ in the function field case, which is an effective divisor of the function field K if additive notation is used). Moreover, for any subset S of $\mathbb{h}_K \cup \mathbb{a}_K$, an S-cycle $\mathfrak{m}$ of K is by definition a cycle $\mathfrak{m}$ of K with support disjoint from S.

If K is assumed to be a number field, let S denote $\mathbb{a}_{K,\mathbb{C}}$. An S-cycle $\mathfrak{m} = \prod_{v\in\mathbb{h}_K\cup\mathbb{a}_K} v^{e_v}$ of K canonically defines a subgroup $\mathscr{U}_{S,\mathfrak{m}}^{\varphi_K}(= \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ in short) of $\mathscr{I}_K^{\varphi_K}$ by the free product

$$\mathscr{U}_{\mathfrak{m}}^{\varphi_K} = *_v\,\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})},$$

where the local groups $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})}$ for $v \in \mathbb{h}_K \cup \mathbb{a}_K$ are defined as follows :

- $v \in \mathbb{h}_K$ and $e_v = 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}}$.
- $v \in \mathbb{h}_K$ and $e_v > 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{e_v}}}$.
- $v \in \mathbb{a}_{K,\mathbb{R}}$ and $e_v = 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = W_{\mathbb{R}}$.
- $v \in \mathbb{a}_{K,\mathbb{R}}$ and $e_v = 1$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = W_{\mathbb{R},>0}$, where $W_{\mathbb{R},>0}$ is the subgroup of $W_{\mathbb{R}}$ which is defined as the pre-image of the subgroup $\mathbb{R}_{>0}$ of $\mathbb{R}^\times$ under the natural abelianization homomorphism $W_{\mathbb{R}} \to W_{\mathbb{R}}^{ab} \xrightarrow{\sim} \mathbb{R}^\times$ of $W_{\mathbb{R}}$.
- $v \in \mathbb{a}_{K,\mathbb{C}}$: So $e_v = 0$ and we set $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = W_{\mathbb{C}}$.

In case K is a function field with full constant field $\mathbb{F}_q$, following [1], S denotes *any* fixed subset of places of K and an S-cycle $\mathfrak{m} = \prod_{v\in\mathbb{h}_K} v^{e_v}$ of K is nothing but an effective divisor with support disjoint from S in additive notation. Moreover, assume that $S \neq \varnothing$ (look at Section 1 of [1]). Then the S-cycle $\mathfrak{m}$ of K canonically defines a subgroup $\mathscr{U}_{S,\mathfrak{m}}^{\varphi_K} (= \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ in short) of $\mathscr{J}_K^{\varphi_K}$ by

$$\mathscr{U}_{\mathfrak{m}}^{\varphi_K} = \left(*_v \mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} \right) \cap \mathscr{J}_K^{\varphi_K},$$

where the local groups $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})}$ for $v \in \mathbb{h}_K$ are defined as follows :

- $v \in \mathbb{h}_K - S$ and $e_v = 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}}$.
- $v \in \mathbb{h}_K - S$ and $e_v > 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{e_v}}}$.
- $v \in S$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$.

For the definition and the basic properties of the subgroup ${}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{i}}}$ of ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$, where $\underline{i}$ is an "increasing net" in $\mathbb{R}_{\geq -1}$, look at [7].

Now, assume that Conjecture 3.2 holds in this subsection.

Theorem 4.25. *For any S-cycle $\mathfrak{m}$ of K,*

$$\mathscr{N}_K^{\varphi_K} \mathscr{J}_K^{\varphi_K^c} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K} = \mathscr{N}_K^{\varphi_K} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$$

is an open normal subgroup of finite index in $\mathscr{J}_K^{\varphi_K}$ containing $\mathscr{N}_K^{\varphi_K}$, where

$$\mathsf{s}_K : \mathscr{J}_K^{\varphi_K} \twoheadrightarrow \mathbb{J}_K$$

is the continuous surjective homomorphism defined by the abelianization map of $\mathscr{J}_K^{\varphi_K}$.

Proof. The inclusion[7] $\mathscr{J}_K^{\varphi_K^c} \subseteq \mathsf{s}_K^{-1}(K^\times)$ is clear. Under the arrow $\mathsf{s}_K : \mathscr{J}_K^{\varphi_K} \twoheadrightarrow \mathbb{J}_K$, the subgroup $\mathscr{J}_K^{\varphi_K^c} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ of $\mathscr{J}_K^{\varphi_K}$ is mapped onto the subgroup $K^\times U_{\mathfrak{m}}$ of $\mathbb{J}_K$. Here, $U_{\mathfrak{m}}$ is the subgroup of $\mathbb{J}_K$ canonically defined by the cycle $\mathfrak{m}$. Moreover, $\mathscr{J}_K^{\varphi_K^c} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ is the pre-image of $K^\times U_{\mathfrak{m}}$ under the natural surjection $\mathsf{s}_K : \mathscr{J}_K^{\varphi_K} \twoheadrightarrow \mathbb{J}_K$. As $K^\times U_{\mathfrak{m}}$ is a finite index open subgroup of $\mathbb{J}_K$ containing $K^\times$ (if K is assumed to be a function field, then we apply Proposition 1.1 of [1]), it follows that $\mathscr{J}_K^{\varphi_K^c} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ is a finite index open and normal subgroup of $\mathscr{J}_K^{\varphi_K}$. Therefore, $\mathscr{N}_K^{\varphi_K} \mathscr{J}_K^{\varphi_K^c} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ is an open normal subgroup of finite index in $\mathscr{J}_K^{\varphi_K}$ containing $\mathscr{N}_K^{\varphi_K}$, which completes the proof. □

Given any S-cycle $\mathfrak{m} = \prod_{v \in \mathbb{h}_K \cup \mathbb{a}_K} v^{e_v}$ of K. By Theorem 4.21 of Subsection 4.8 on the *non-abelian global existence*, there exists a finite Galois extension $R_{\mathfrak{m}}$ of K inside K^{sep}, called the *S-ray class field of* $\mathfrak{m}$, satisfying

$$\mathsf{NR}_{R_{\mathfrak{m}}/K}^{\varphi_K \ \mathrm{Weil}^*} : \mathscr{J}_K^{\varphi_K} / \mathscr{N}_K^{\varphi_K} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K} \xrightarrow{\sim} \mathrm{Gal}(R_{\mathfrak{m}}/K).$$

The group $\mathscr{J}_K^{\varphi_K} / \mathscr{N}_K^{\varphi_K} \mathsf{s}_K^{-1}(K^\times) \mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ is called the *S-ray class group of* $\mathfrak{m}$.

Moreover, we have the following two theorems about the Galois extension $R_{\mathfrak{m}}/K$, where $\mathfrak{m} = \prod_{v \in \mathbb{h}_K \cup \mathbb{a}_K} v^{e_v}$ is an S-cycle of K.

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[7] In fact, if $\mathsf{s}_K = \mathsf{a}_K$, then $\mathscr{J}_K^{\varphi_K^c} = \mathsf{s}_K^{-1}(K^\times)$ by Theorem 4.3.

Theorem 4.26. *The Galois extension $R_{\mathfrak{m}}$ over K is unramified at all v with $e_v = 0$.*

Proof. Let $v \in \mathbb{h}_K$. The extension $R_{\mathfrak{m}}/K$ is unramified at v if and only if the inertia group $I_v := \mathrm{I}_v(R_{\mathfrak{m}}/K)$ of v in $\mathrm{Gal}(R_{\mathfrak{m}}/K)$ determined by the continuous homomorphism $e_\mu^{\mathrm{Weil}} : W_{K_v} \to W_K$ is trivial. Recall that, the subgroup I_v of $\mathrm{Gal}(R_{\mathfrak{m}}/K)$ is defined in Subsection 4.6 by

$$I_v = \mathrm{res}^*_{R_{\mathfrak{m}}} \circ \beta_{R_{\mathfrak{m}}/K} \circ \mathrm{red}_{W_{R_{\mathfrak{m}}}} \circ e_v^{\mathrm{Weil}}(W^0_{K_v}).$$

Now, assume furthermore that, $e_v = 0$. By Theorem 4.17 combined with the fact that $q_v({}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}}) \subset \ker(\mathsf{NR}^{\underline{\varphi_K}}_{R_{\mathfrak{m}}/K}{}^{\mathrm{Weil}})$, it follows that

$$I_v = \mathsf{NR}^{\underline{\varphi_K}}_{R_{\mathfrak{m}}/K}{}^{\mathrm{Weil}} \circ q_v({}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}}) = 1,$$

which proves that $R_{\mathfrak{m}}/K$ is unramified at such v. Next, let $v \in \mathbb{a}_K$ and assume that $e_v = 0$. Then $I_v = 1$ by Theorem 4.17 combined with the fact that $q_v(W^0_{K_v}) \subset \ker(\mathsf{NR}^{\underline{\varphi_K}}_{R_{\mathfrak{m}}/K}{}^{\mathrm{Weil}})$, where $W^0_{K_v} = W_{K_v}$. □

Let L be any finite Galois extension of K in K^{sep}. By the non-abelian existence theorem (Theorem 4.21 of Subsection 4.8), the open normal subgroup of finite index in $\mathscr{J}_K^{\underline{\varphi_K}}$ containing $\mathscr{N}_K^{\underline{\varphi_K}}$ and corresponding to L is $\mathscr{N}_K^{\underline{\varphi_K}} \mathscr{N}^\infty_{L/K}(\mathscr{J}_L^{\underline{\varphi_L}})$. Therefore, the subgroup $\mathsf{s}_K^{-1}(K^\times)\mathscr{N}_K^{\underline{\varphi_K}} \mathscr{N}^\infty_{L/K}(\mathscr{J}_L^{\underline{\varphi_L}})$ of $\mathbb{J}_K$ is open and of finite index in $\mathbb{J}_K$, and corresponds to a subfield $L_!$ of L under the non-abelian existence theorem.

Theorem 4.27. *Let L be any finite Galois extension of K in K^{sep}. Then $L_!$ is a subfield of $R_{\mathfrak{m}}$ for some cycle $\mathfrak{m}$ of K.*

Proof. Let L be a finite Galois extension of K. The subgroup $\mathsf{s}_K(\mathscr{N}_K^{\underline{\varphi_K}} \mathscr{N}^\infty_{L/K}(\mathscr{J}_L^{\underline{\varphi_L}}))$ of $\mathbb{J}_K$ is open and of finite index in $\mathbb{J}_K$. Thus, there exists a cycle $\mathfrak{m}$ of K such that

$$K^\times U_{\mathfrak{m}} \subseteq K^\times \mathsf{s}_K(\mathscr{N}_K^{\underline{\varphi_K}} \mathscr{N}^\infty_{L/K}(\mathscr{J}_L^{\underline{\varphi_L}})),$$

where $U_{\mathfrak{m}}$ is the subgroup of $\mathbb{J}_K$ canonically defined by $\mathfrak{m}$. Hence, the following inclusions hold :

$$\mathsf{s}_K^{-1}(K^\times)\mathscr{U}_{\mathfrak{m}}^{\underline{\varphi_K}} \subseteq \mathscr{N}_K^{\underline{\varphi_K}} \mathsf{s}_K^{-1}(K^\times)\mathscr{U}_{\mathfrak{m}}^{\underline{\varphi_K}} \subseteq \mathsf{s}_K^{-1}(K^\times)\mathscr{N}_K^{\underline{\varphi_K}} \mathscr{N}^\infty_{L/K}(\mathscr{J}_L^{\underline{\varphi_L}}).$$

Thus, by the non-abelian existence theorem and by Theorem 4.25,

$$L_! \subseteq R_{\mathfrak{m}},$$

which completes the proof. □

4.10. The set of primes in K that split in a finite extension L/K

Let L be a finite Galois extension of a global field K. Denote the set of finite primes v in K that split completely in L by $\mathrm{Spl}(L/K)$. The aim of this section is to characterize the set $\mathrm{Spl}(L/K)$ in terms of the base field K alone. We assume that Conjecture 3.2 holds in this subsection.

Recall that, a finite prime v in K splits completely in the Galois extension L over K if and only if the decomposition group $\mathrm{D}_v(L/K)$ of v in $\mathrm{Gal}(L/K)$ is trivial.Thus, by Theorem 4.17 combined with Theorem 4.14, the following theorem follows immediately.

Theorem 4.28. *A prime ν in K splits completely in L/K if and only if* $q_\nu({}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}) \subset \ker(\mathsf{NR}_{L/K}^{\varphi_K\ \mathrm{Weil}}) = \mathscr{N}_K^{\varphi_K}\mathscr{N}_{L/K}^{\infty}(\mathscr{J}_L^{\varphi_L})$.

Thus, by Theorem 4.28, the set $\mathrm{Spl}(L/K)$ is characterized in terms of the base field K by

$$\mathrm{Spl}(L/K) = \{\nu \in \mathbb{h}_K \mid q_\nu({}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}) \subset \ker(\mathsf{NR}_{L/K}^{\varphi_K\ \mathrm{Weil}})\}.$$

One final remark is in order.

Remark 4.29. In case L/K is a finite abelian extension, then by Lemma 4.1 and Theorem 4.16 applied to the abelian extension L/K,

$$\begin{aligned}\mathrm{Spl}(L/K) &= \{\nu \in \mathbb{h}_K \mid q_\nu({}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}) \subset \ker(\mathsf{NR}_{L/K}^{\varphi_K\ \mathrm{Weil}})\} \\ &= \{\nu \in \mathbb{h}_K \mid \varepsilon_\nu(K_\nu^\times) \subset \ker(\bullet, L/K)\},\end{aligned}$$

which is nothing but the well-known formulation of the set of primes in K that split completely in the abelian extension L/K.

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