On the Classification of Integrable Scalar Evolution Equations in 1 + 1 Dimension

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October 5, 2010
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Aim: Identify “Integrable Equations”

- Scalar equations in one space dimension $u = u(x,t)$; Notation: $u_k = \frac{\partial^k u}{\partial x^k}$.

- Evolution equations: $u_t = F(x,t,u,u_1,\ldots,u_m)$, $m$ is fixed but arbitrary. (Notation: $F_k = \frac{\partial F}{\partial u_k}$).

We use the method proposed by Mikhailov, Shabat and Sokolov, in, What is Integrability? Springer Series in Nonlinear Dynamics, Ed. V.E. Zakharov; 1991.
Two types of “integrable” equations:

- Linear equations or those equations that can be transformed to a linear equation by a differential substitution are called \( C \)-integrable (Change of variable); The prototype is the Burger’s equation \( u_t = u_{xx} + uu_x \), transformed to the heat equation \( v_t = v_{xx} \), by the Cole-Hopf transformation: \( u = 2v_x/v \).

- Nonlinear equations that can be solved by the “Inverse Spectral Transform” are \( S \)-integrable.
$S$-integrable equations in 1 space dimension

- The Korteweg-deVries (KdV) equation
  \[ u_t = u_{xxx} + uu_x \]
  (solved by the “inverse spectral transformation” / “inverse scattering” method in 1967 (Gardner, Greene, Kruskal, Miura).

- The Sawada-Kotera equation (1974)
  \[ u_t = u_{xxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2u_x \]

- The Kaup-Kupershmidt equation (1980)
  \[ u_t = u_{xxxx} + 5uu_{xxx} + 25/2u_x u_{xx} + 5u^2u_x \]

- 5th order KdV:
  \[ u_t = u_{xxxx} + \beta uu_{xxx} + 2\beta u_x u_{xx} + \frac{3}{10}\beta^2u^2u_x \]
Integrability Tests: Strategy

• Look for properties common to all/most integrable equations,

• Choose a property that will select a restricted group of equations among a general class

• The known integrable equations should be in this restricted class

• We hope that other equations in the restricted class are also “integrable”
Properties of the KdV equation...

- It has an infinite number of “generalized symmetries”, and conserved quantities,

- It has two compatible Hamiltonian structures,

- “Soliton” solutions,

- It can be written as the integrability condition of a linear system (Lax pair),

- Its reduction to ordinary differential equations has no movable critical points (Painleve test).
The search for new integrable equations...

...after the solution of the KdV equation (1967) people looked to find and solve “such equations” ... Only “truly new” equations are the Sawada-Kotera and Kaup equations (1975,1980):

...most equations found were related to known integrable equations: -Equations obtained from the known ones by a Miura type transformation ($u = \varphi(v, v_x)$); modified, potential etc. forms of the original equation.

- The KdV equation $u_t = u_{xxx} + uu_x$,
- The modified KdV equation $v_t = v_{xxx} - \frac{1}{6}v^2v_x$,
- Their Miura transformation: $u = v_x - \frac{1}{6}v^2$. 
Hierarchies of integrable equations: The recursion operator...

- The KdV equation: \( u_t = u_{xxx} + uu_x, \)

- Its recursion operator: \( R = D^2 + \frac{2}{3}u + \frac{1}{3}u_x D^{-1}, \) \( D \) is the total derivative with respect to \( x, \) \( D^{-1} \phi = \phi D^{-1} - D^{-1} \phi_x D^{-1}. \)

- The 5th order KdV equation is given by \( u_t = R(u_x) \)

\[
R(u_t) = R(u_{xxx} + uu_x) = u_{xxxx} + \frac{5}{3} uu_{xxx} + \frac{10}{3} uu_{xx} + \frac{5}{6} u^2 u_x.
\]

- One can obtain similar equations at each odd order, called the “KdV Hierarchy”.
Lax Pairs for KdV, Sawada-Kotera, Kaup equations...

Spectral problem: Find operators $L$, $P$, depending on $u(x,t)$ such that the compatibility of the system

$$L(u)\psi = \lambda \psi, \quad \psi_t = P\psi$$

gives the evolution equation for $u$. ($L$ is a differential operator, $L^{1/n}$ is a formal series, $L^{k/2}_+$ means the differential part)

- KdV equation: 2nd order spectral problem:
  $$ (D^2 + u)\psi = \lambda \psi, \quad \psi_t = P\psi, \quad P = L^{k/2}_+ $$

- Sawada-Kotera, Kaup equations: 3rd order spectral problem:
  $$ (D^3 + auD + bu_1)\psi = \lambda \psi, \quad \psi_t = P\psi, \quad P = L^{k/3}_+ $$

This leads to hierarchies of integrable equations: Gelfand-Dikii flows
Non-existence results for integrable hierarchies: The Wang-Sanders result...

- 1998, ‘main result of the Ph.D thesis of Jing Ping Wang: “Polynomial, scale invariant, scalar evolution equations in 1 space dimension, of order greater than or equal to 7 are symmetries of lower order equations”

- Similar results by Wang and Sanders for equations involving negative powers,

No new equations in the class of scale invariant polynomial equations of order 7 and larger.
Is it possible to have “new” equations? Results:

- Third order equations: Preliminary classification is given in MSS; There are 3 classes; The class of essentially non-linear equations is studied by Svinolupov (classification is not complete, the method suggests replacing the dependent variable by a conserved density).

- Fifth order equations: There may be non quasilinear equations; Quasilinear equations with constant separant (coefficient of $u_5$) are classified in MSS; The classification of quasilinear equations with non-constant separant $a$ is almost complete [Bilge, Ozkum], they are polynomial in $a$.

- Non-polynomial higher order equations: It is proved that equations of order $m \geq 7$ are polynomial in top three derivatives; It is shown that at orders $m = 7, 9, 11$ they are polynomial in $u_k$ for $k > 3$ and the separant $a$ has the same form as the one for order 5 [Bilge, Mizrahi].
Recursion operators and canonical densities.

- A “symmetry” $\sigma$ satisfies $\sigma_t = F_\ast \sigma$, where $F_\ast = \sum_{i=0}^{m} \frac{\partial F}{\partial u_i} D^i$, where $F$ depends on $u_k$ $k = 0, \ldots, m$,

- A “recursion operator” $R$ sends symmetries to symmetries, $(R\sigma)_t = F_\ast (R\sigma)$.

- We work with recursion operators that satisfy $R_t + [R, F_\ast] = 0$ ,

- We can expand $R$ in a Laurent series in $D^{-1}$, $D = \frac{d}{dx}$, $R = R_{-k}D^k + \ldots + R_1D^{-1} + \ldots$,

- If $R$ is a recursion operator of order $k$, $R^{n/k}$ is also a recursion operator,

- The coefficients of $D^{-1}$ in $R^{n/k}$ are called the canonical densities,

- If $R$ is a recursion operator of order $m$ it is $R = F_\ast + L$ where $L$ has order 1.
Outline of the derivation:

- Compute the formal series expansion of the first order recursion operator $R$ for arbitrary order $m$.
- The coefficients of $D^{-1}$ in $R^k$ are conserved quantities, called the “canonical densities”.
- The conserved densities are at most quadratic in the highest derivatives.
- The conserved density conditions are obtained with computer algebra.
- Equations of order 5 appear as an exception.
- We prove that equations of orders $m \geq 7$ are polynomial in top 3 derivatives.
- Classification of quasilinear 5th order equations are almost complete.
- Lower order equations ($m = 7, 9, 11, ..$) are polynomial in $u_k$ for $k > 3$, their dependency to lower order derivatives are similar to order 5.
Notation:  \( F_m = \frac{\partial F}{\partial u_m}, \)  \( F_{m-1} = \frac{\partial F}{\partial u_{m-1}}, \)  \( a = F_m^{1/m}, \)  \( \alpha(i) = \frac{F_{m-i}}{F_m}, \)  \( i = 1, 2, 3, 4. \)

If \( u_t = F[u] \) is integrable, then  \( \rho^{(-1)} = F_{m-1}^{1/m}, \)  \( \rho^{(0)} = F_{m-1}/F_m, \) are conserved densities for equations of any order.

**Higher Order Conserved Densities (UP TO TOTAL DERIVATIVES):**

\[
\rho^{(1)} = a^{-1}(Da)^2 - \frac{12}{m(m+1)}Da\alpha(1) + a \left[ \frac{12}{m^2(m+1)}\alpha^2(1) - \frac{24}{m(m^2-1)}\alpha(2) \right],
\]

\[
\rho^{(2)} = a(Da) \left[ D\alpha(1) + \frac{3}{m}\alpha^2(1) - \frac{6}{(m-1)}\alpha(2) \right] + 2a^2 \left[ -\frac{1}{m^2}\alpha^3(1) + \frac{3}{m(m-1)}\alpha(1)\alpha(2) - \frac{3}{(m-1)(m-2)}\alpha(3) \right],
\]
\[
\rho^{(3)} = a(D^2 a)^2 - \frac{60}{m(m+1)(m+3)} a^2 D^2 a D\alpha_{(1)} + \frac{1}{4} a^{-1} (Da)^4 \\
+ 30a(Da)^2 \left[ \frac{(m-1)}{m(m+1)(m+3)} D\alpha_{(1)} + \frac{1}{m^2(m+1)} \alpha_{(1)}^2 - \frac{2}{m(m^2-1)} \alpha_{(2)} \right] \\
+ \frac{120}{m(m^2-1)(m+3)} a^2 Da \left[ -\frac{(m-1)(m-3)}{m} \alpha_{(1)} D\alpha_{(1)} + (m-3) D\alpha_{(2)} \\
- \frac{(m-1)(2m-3)}{m^2} \alpha_{(1)}^3 + \frac{6(m-2)}{m} \alpha_{(1)} \alpha_{(2)} - 6\alpha_{(3)} \right] \\
+ \frac{60}{m(m^2-1)(m+3)} a^3 \left[ \frac{(m-1)}{m} (D\alpha_{(1)})^2 - \frac{4}{m} Da \alpha_{(1)} \alpha_{(2)} + \frac{(m-1)(2m-3)}{m^3} \alpha_{(1)}^4 \\
- \frac{4(2m-3)}{m^2} \alpha_{(1)}^2 \alpha_{(2)} + \frac{8}{m} \alpha_{(1)} \alpha_{(3)} + \frac{4}{m} \alpha_{(2)}^2 - \frac{8}{(m-3)} \alpha_{(4)} \right].
\]

We compute \(D_t \rho\), integrate by parts, until we obtain a term which is nonlinear in its highest derivative: The coefficient of this term should be zero. This gives partial differential equations that determine \(F\).
First Result: Quasilinearity

**Theorem** Let \( u_t = F[u] \) be a scalar evolution equation of \( m = 2k + 1 \) where \( k \geq 3 \), admitting a nontrivial conserved density

\[
\rho = P u_n^2 + Q u_n + R
\]

of order \( n = m + 1 \), where \( P \), \( Q \) and \( R \) are independent of \( u_m \). Then

\[
 u_t = A u_m + B,
\]

where \( A \) and \( B \) are independent of \( u_m \). [Bilge, 2005]

The canonical density \( \rho^{(1)} \) is of the form above, hence evolution equations of order greater than 5 are quasi-linear.
Second Result: Polynomiality in top 3 derivatives

Theorem Let $u_t = F[u]$ be a scalar evolution equation of $m = 2k + 1$ where $k \geq 3$, admitting the canonical conserved densities $\rho^{(i)}$, $i = 1, 2, 3$. Then

$$u_t = (Au_m + Bu_{m-1}u_{m-2} + Cu_{m-2}^3) + (Eu_{m-1} + Gu_{m-2}^2) + (Hu_{m-2}) + (K)$$

where $A, B, \ldots, K$ depend on $x, t, u, \ldots, u_{m-3}$.

[Bilge, Mizrahi 2008]
Third Result: 5th order quasilinear equations with non-constant separant:

**Theorem** Let \( u_t = F[u] \), where \( F \) depends on \( x, t, u, \ldots, u_5 \) and assume that \( F \) is quasilinear. Then

\[
 u_t = a^5u_5 + Bu_4^2 + Cu_4 + G 
\]

- If all conserved are nontrivial,
  \[
  a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2}
  \]

- If \( \rho_3 \) is trivial
  \[
  a = (\lambda u_3 + \mu)^{-1/3}
  \]

where \( \alpha, \beta, \gamma, \lambda, \mu \) are functions of \( x, t, u, u_1, u_2 \).
Forth Result: “Level homogeneous” equations of order 7, 9, 11 admidding “level homogeneous” conserved densities

• They are polynomial in $u_k$ for $k \geq 4$

• In fact, it can be proved that if the equation is “level homogeneous”, then the conserved densities are also level homogeneous.

• It needs to be proved that if it is polynomial and level homogeneous with respect to $u_k$, and the separant is independent of $u_{k-1}$, the equation will be polynomial and level homogeneous with respect to $u_{k-1}$ (MAIN STEP)

• It has been shown that

$$u_t = (\alpha u_3^2 + \beta u_3 + \gamma)^{-m/2} u_m + B u_4 u_{m-1} + \ldots$$

That is the separant is the same as for 5th order equations
Remarks

- Essentially non-linear classes of integrable equations arising at the third order are absent in higher orders!

- Although dependencies in all variables were used in the derivations, the equations relevant for obtaining polynomiality results involved the (non-polynomial) dependencies on top derivatives (First on $u_m$ only, then on $u_{m-1}$ only, and on $u_{m-2}$ only at the last step).

- This observation led to the definition of a new type of grading on differential polynomials that we called “level above $k$”;
Why 5th order is an exception to quasilinearity?

We start with \( u_t = F \), use the existence of a conserved density \( \rho^{(1)} \sim P u_{m+1}^2 \), \( m = 2k + 1 \) and \( l = 1 \) to get the homogeneous linear system

\[
\begin{pmatrix}
u_0(k,l) & u_1(k,l) \\
A_0(k,l) & D_0(k) + K_0(k)
\end{pmatrix}
\begin{pmatrix}
P F'' \\
P' F'
\end{pmatrix}
= \begin{pmatrix}0 \\
0\end{pmatrix}
\]

The coefficient matrix is nonsingular for \( k \neq 2 \), that is except for 5th order equations. The term \( P \) is in fact nonzero for the first canonical density, hence \( F''' = 0 \). This proves quasilinearity. Here prime is the derivative with respect to top derivative \( u_{m-1} \). Hence

\[
u_t = A(x,t,u,\ldots,u_{m-1})u_m + B(x,t,u,\ldots,u_{m-1})
\]

The final result is:

\[
u_t = A u_m + B u_{m-1} u_{m-2} + C u_{m-2}^3 + E u_{m-1} + G u_{m-2}^2 + H u_{m-2} + K
\]

where the coefficients depend on at most \( u_{m-3} \).

This suggests a new type of scaling...
The grading by “Level above $k$”  Set up: Functions involved are polynomial in $u_{k+1}, \ldots$ but have arbitrary functional dependency on the variables $x,t,u,\ldots,u_k$.

Example: If $\varphi = \varphi(x,t,u,\ldots,u_k)$, then

\[
D\varphi = \frac{\partial \varphi}{\partial u_k}u_{k+1} + \alpha
\]

\[
D^2\varphi = \frac{\partial \varphi}{\partial u_k}u_{k+2} + \frac{\partial^2 \varphi}{\partial u_k^2}u_{k+1}^2 + \beta u_{k+1} + \gamma
\]

\[
D^3\varphi = \frac{\partial \varphi}{\partial u_k}u_{k+3} + 3\frac{\partial^2 \varphi}{\partial u_k^2}u_{k+1}u_{k+2} + \frac{\partial^3 \varphi}{\partial u_k^3}u_{k+1}^3 + \mu u_{k+2} + \nu u_{k+1}^2 + \lambda u_{k+1} + \eta
\]

where $\alpha, \beta$ etc. depend on at most $u_k$.

We define the “level above $k$” as the total number of derivatives above $k$. 
Examples of level homogenous equations:

Mikhailov-Shabat-Sokolov: Classification of fifth order equations

\[ u_t = u_5 + f(x, t, u, u_1, u_2, u_3, u_4). \]

Up to dependencies on \( x, t, u, u_1 \), they obtain the form

\[ u_t = u_5 + (A_1 u_2 + A_2) u_4 + A_3 u_3^2 + (A_4 u_2^2 + A_5 u_2 + A_6) u_3 + A_7 u_2^4 + A_8 u_2^3 + A_9 u_2^2 + A_{10} u_2 + A_{11}. \]

Rearranging,

\[
\begin{align*}
    u_t &= u_5 + A_1 u_2 u_4 + A_3 u_3^2 + A_4 u_2^2 u_3 + A_7 u_2^4 \\
    &\quad + A_2 u_4 + A_5 u_2 u_3 + A_8 u_2^3 \\
    &\quad + A_6 u_3 + A_9 u_2^2 + A_{10} u_2 + A_{11}.
\end{align*}
\]

At fifth order, level homogeneity stops here, one needs to use differential substitutions to simplify and recover level homogeneous expressions.
Why we use this grading?

Level above \( k \) is invariant under integration by parts.: Let \( p_1 < p_2 < \cdots < p_l < s - 1 \). Then

\[
\begin{align*}
\varphi u_{p_1}^{a_1} \cdots u_{p_l}^{a_l} u_s &\equiv -D \left( \varphi u_{p_1}^{a_1} \cdots u_{p_l}^{a_l} \right) u_{s-1}, \\
\varphi u_{p_1}^{a_1} \cdots u_{p_l}^{a_l} p_{s-1} u_s &\equiv -\frac{1}{p+1} D \left( \varphi u_{p_1}^{a_1} \cdots u_{p_l}^{a_l} \right) p_{s-1}^{p+1}.
\end{align*}
\]

The integration by parts is repeated until one encounter a non-integrable monomial as:

\[
u_{p_1}^{a_1} \cdots u_{p_l}^{a_l} u_s, \quad p > 1.
\]

In symbolic computation: Top level terms depend on the top derivative only. Use only the dependency on the top derivative.

In determining the form of canonical densities: If the evolution equation is level homogeneous above \( k \), its recursion operator is also level homogeneous. If the canonical densities are nontrivial, there are level homogeneous conserved densities (proved).
If $F$ is level homogeneous above every base level, how do we write $F$?

This uses the “decomposition of integers”

<table>
<thead>
<tr>
<th>Base $u_k$</th>
<th>$u_{m-1}$</th>
<th>$u_{m-2}$</th>
<th>$u_{m-3}$</th>
<th>$u_{m-4}$</th>
<th>$u_{m-5}$</th>
<th>$u_{m-6}$</th>
<th>$u_{m-7}$</th>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$u_{m-2}$</td>
<td>2</td>
<td>1+1</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
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<td>2+1</td>
<td>1+1+1</td>
<td></td>
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<td></td>
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</tr>
<tr>
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<td>2+1+1</td>
<td>1+1+1+1</td>
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<td>3+1+1</td>
<td>2+1+1+1</td>
<td>1+1+1+1+1</td>
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</tr>
<tr>
<td>$u_{m-6}$</td>
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<td>4+1+1</td>
<td>3+1+1+1</td>
<td>2+1+1+1+1</td>
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<td>6 x 1</td>
</tr>
<tr>
<td>$u_{m-7}$</td>
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<td>6+1</td>
<td>5+1+1</td>
<td>4+1+1+1</td>
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<td>2+1+1+1+1</td>
</tr>
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</table>
7th order equations: Top level terms above each level.

<table>
<thead>
<tr>
<th>Base</th>
<th>Level</th>
<th>Term 1</th>
<th>Term 2</th>
<th>Term 3</th>
<th>Term 4</th>
<th>Term 5</th>
<th>Term 6</th>
<th>Term 7</th>
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<td>$u_1$</td>
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<td>$u_4u_2u_2$</td>
<td>$u_3^3 u_2 u_2$</td>
<td>$u_2^2 u_2 u_1 u_1$</td>
<td>$u_1^2 u_1$</td>
</tr>
<tr>
<td>$u$</td>
<td>7</td>
<td>$u_7$</td>
<td>$u_6u_1$</td>
<td>$u_5u_2u_1$</td>
<td>$u_4u_1$</td>
<td>$u_3^4 u_1$</td>
<td>$u_2^5 u_1$</td>
<td>$u_1^7$</td>
</tr>
</tbody>
</table>
How can we use level homogeneity?

- We have proved level homogeneity in top three derivatives,

- We have proved that if the equation is level homogeneous and if it admits a recursion operator, the operator is also level homogeneous, provided that the separant is of level zero (this involves solutions of first order ODE’s).

- It follows that the canonical densities are level homogeneous.

- The canonical densities may or may not be trivial. If they are trivial, this gives PDE’s for \( F \), in \( u_t = F \). For example, the triviality of \( \rho_3 \) leads to the Sawada-Kotera, Kaup equations.

- If all conserved densities are nontrivial we expect to obtain hierarchies related to the KdV.
• MAIN PROBLEM: Prove in general that level homogeneity above a base $k$ and the existence of conserved densities lead to level homogeneity above the base level $k - 1$. This involves non-singularity of systems of algebraic equations at the top level,

• But for lower levels, we have to solve differential equations; their solutions may a priori involve logarithms and algebraic functions, but the coefficients of such terms vanish (observed at order 5).

• The whole scheme breaks down when $\partial a/\partial u_k$ is nonzero. This is the case $k = 3$

• After this stage we have arbitrary functions, we should find transformations to eliminate them.
How to explain missing conserved densities?

• The Korteweg-deVries hierarchy have a recursion operator of order 2 and has a starting symmetry $u_x$, hence there are symmetries, conserved covariants (co-symmetries) and conserved densities at every other order. Another interpretation is that the Lax operator is of order 2 and even order symmetries are missing.

• The Sawada-Kotera and Kaup hierarchies have recursion operators of order 6 with two starting symmetries of orders 1 and 5, hence it has symmetries at orders $1 + 6k$ and $5 + 6k$. An alternative interpretation is that the Lax operator has order 3 and symmetries are missing at orders that are multiples of 3.
Can we use transformations involving higher order derivatives?

Given \( u_t = F(x,t,u,\ldots,u_m) \), can we set

\[
\frac{\partial F}{\partial u_m} = 1, \quad \text{and} \quad \frac{\partial F}{\partial u_{m-1}} = 0.
\]

If \( D_t \rho = D_x \eta \) and \( \rho \) is nontrivial, then

\[
dx' = \rho(u,u_1)dx + \eta(u,u_1,\ldots,u_n)dt,
\]

\( t' = t, \ u' = \psi(u), \ u'_k = \left(\frac{1}{\rho}D\right)^k \psi, \ k = 1,2,\ldots \) [MSS, p.127] defines a locally an invertible transformation.

- Consider \( u_t = A(u,u_1)u_m + B(u,u_1,\ldots,u_{m-1}) \).
- \( \rho_{-1} = A^{-1/m} \) is always a conserved density,
- Use this to set \( A = 1 \).

If \( A \) depends on higher derivatives, the transformed equation may not be local! (This is our main problem now)
What’s next?

- Scale invariance/level homogeneity is a consequence of the existence of the recursion operator,

- Can we set $A = 1$ by a differential transformation and show that integrability guarantees that the resulting equation is local?
The form of the exponents $-1/2$ and $-1/3$ in the separant $a$ suggests that they are related to 2nd and 3rd order Lax operators.
Thank you...


P.J. Olver, it Evolution equations possessing infinitely many symmetries, (Springer-Verlag, Berlin 1993)