On the Classification of Integrable Scalar Evolution Equations in $1+1$ Dimension

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Aim: Identify "Integrable Equations"

- Scalar equations in one space dimension $u = u(x,t)$; Notation: $u_k = \frac{\partial^k u}{\partial x^k}$,

- Evolution equations: $u_t = F(x,t,u,u_1,\ldots,u_m)$, m is fixed but arbitrary. (Notation: $F_k = \frac{\partial F}{\partial u_k}$).

We use the method proposed by Mikhailov, Shabat and Sokolov, in, What is Integrability? Springer Series in Nonlinear Dynamics, Ed. V.E. Zakharov; 1991.

Two types of “integrable” equations:

- Linear equations or those equations that can be transformed to a linear equation by a differential substitution are called C-integrable (Change of variable); The prototype is the Burger’s equation $u_t = u_{xx} + uu_x$, transformed to the heat equation $v_t = v_{xx}$, by the Cole-Hopf transformation: $u = 2v_x/v$.

- Nonlinear equations that can be solved by the “Inverse Spectral Transform” are S-integrable.

S-integrable equations in 1 space dimension

- The Korteweg-deVries (KdV) equation

$$u_t = u_{xxx} + uu_x$$

  (solved by the "inverse spectral transformation" / "inverse scattering" method in 1967 (Gardner, Greene, Kruskal,Miura).

- The Sawada-Kotera equation (1974)

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x$$

- The Kaup-Kuppershmidt equation (1980)

$$u_t = u_{xxxxx} + 5uu_{xxx} + 25/2 u_x u_{xx} + 5u^2 u_x$$

- 5th order KdV:

$$u_t = u_{xxxxx} + \beta uu_{xxx} + 2\beta u_x u_{xx} + \frac{3}{10}\beta^2 u^2 u_x$$

Integrability Tests: Strategy

- Look for properties common to all/most integrable equations,
- Choose a property that will select a restricted group of equations among a general class
- The known integrable equations should be in this restricted class
- We hope that other equations in the restricted class are also "integrable"

Properties of the KdV equation...

- It has an infinite number of “generalized symmetries”, and conserved quantities,
- It has two compatible Hamiltonian structures,
- “Soliton” solutions,
- It can be written as the integrability condition of a linear system (Lax pair),
- Its reduction to ordinary differential equations has no movable critical points (Painleve test).

The search for new integrable equations...

...after the solution of the KdV equation (1967) people looked to *find* and *solve* "such equations" ... Only "truly new" equations are the Sawada-Kotera and Kaup equations (1975,1980):

...most equations found were related to known integrable equations: -Equations obtained from the known ones by a Miura type transformation ($u = \varphi(v, v_x)$); modified, potential etc. forms of the original equation.

- The KdV equation $u_t = u_{xxx} + uu_x$,
- The modified KdV equation $v_t = v_{xxx} - \frac{1}{6}v^2 v_x$,
- Their Miura transformation: $u = v_x - \frac{1}{6}v^2$.

Hierarchies of integrable equations: The recursion operator...

- The KdV equation: $u_t = u_{xxx} + uu_x$,

- Its recursion operator: $R = D^2 + \frac{2}{3}u + \frac{1}{3}u_x D^{-1}$, D is the total derivative with respect to x, $D^{-1}\phi = \phi D^{-1} - D^{-1}\phi_x D^{-1}$.

- The 5th order KdV equation is given by $u_t = R(u_x)$

$$R(u_t) = R(u_{xxx} + uu_x) = u_{xxxxx} + \frac{5}{3}uu_{xxx} + \frac{10}{3}u_x u_{xx} + \frac{5}{6}u^2 u_x.$$

- One can obtain similar equations at each odd order, called the "KdV Hierarchy".

Lax Pairs for KdV, Sawada-Kotera, Kaup equations...

Spectral problem: Find operators L, P, depending on $u(x,t)$ such that the compatibility of the system

$$L(u)\psi = \lambda\psi, \quad \psi_t = P\psi$$

gives the evolution equation for u. (L is a differential operator, $L^{1/n}$ is a formal series, $L_+^{k/2}$ means the differential part)

- KdV equation: 2nd order spectral problem:

$$(D^2 + u)\psi = \lambda\psi, \quad \psi_t = P\psi, \quad P = L_+^{k/2}$$

- Sawada-Kotera, Kaup equations: 3rd order spectral problem:

$$(D^3 + auD + bu_1)\psi = \lambda\psi, \quad \psi_t = P\psi, \quad P = L_+^{k/3}$$

This leads to hierarchies of integrable equations: Gelfand-Dikii flows

Non-existence results for integrable hierarchies: The Wang-Sanders result...

- 1998, ‘main result of the Ph.D thesis of Jing Ping Wang: *“Polynomial, scale invariant, scalar evolution equations in 1 space dimension, of order greater than or equal to 7 are symmetries of lower order equations”*

- Similar results by Wang and Sanders for equations involving negative powers,

No new equations in the class of scale invariant polynomial equations of order 7 and larger.

Is it possible to have "new" equations? Results:

- Third order equations: Preliminary classification is given in MSS; There 3 classes; The class of essentially non-linear equations is studied by Svinolupov (classification is not complete, the method suggests replacing the dependent variable by a conserved density).

- Fifth order equations: There may be non quasilinear equations; Quasilinear equations with constant separant (coefficient of u_5) are classified in MSS; The classification of quasilinear equations with non-constant separant a is almost complete [Bilge, Ozkum], they are polynomial in a.

- Non-polynomial higher order equations: It is proved that equations of order $m \geq 7$ are polynomial in top three derivatives; It is shown that at orders $m = 7, 9, 11$ they are polynomial in u_k for $k > 3$ and the separant a has the same form as the one for order 5 [Bilge, Mizrahi].

Recursion operators and canonical densities..

- A "symmetry" σ satisfies $\sigma_t = F_*\sigma$, where $F_* = \sum_{i=0}^{m} \frac{\partial F}{\partial u_i} D^i$, where F depends on u_k $k = 0, \dots, m$,

- A "recursion operator" R sends symmetries to symmetries, $(R\sigma)_t = F_*(R\sigma)$.

- We work with recursion operators that satisfy $R_t + [R, F_*] = 0$,

- We can expand R in a Laurent series in D^{-1}, $D = d/dx$, $R = R_{-k}D^k + \dots + R_1 D^{-1} + \dots$,

- If R is a recursion operator of order k, $R^{n/k}$ is also a recursion operator,

- The coefficients of D^{-1} in $R^{n/k}$ are called the canonical densities,

- If R is a recursion operator of order m it is $R = F_* + L$ where L has order 1.

Outline of the derivation:

- Compute the formal series expansion of the first order recursion operator R for arbitrary order m.
- The coefficients of D^{-1} in R^k are conserved quantities, called the "canonical densities".
- The conserved densities are at most quadratic in the highest derivatives.
- The conserved density conditions are obtained with computer algebra.
- Equations of order 5 appear as an exception.
- We prove that equations of orders $m \geq 7$ are polynomial in top 3 derivatives
- Classification of quasilinear 5th order equations are almost complete
- Lower order equations $(m = 7, 9, 11, ..)$ are polynomial in u_k for $k > 3$, their dependency to lower order derivatives are similar to order 5

Notation: $F_m = \frac{\partial F}{\partial u_m}$, $F_{m-1} = \frac{\partial F}{\partial u_{m-1}}$, $a = F_m^{1/m}$, $\alpha_{(i)} = \frac{F_{m-i}}{F_m}$, $i = 1, 2, 3, 4.$

If $u_t = F[u]$ is integrable, then $\rho^{(-1)} = F_m^{-1/m}$, $\rho^{(0)} = F_{m-1}/F_m$, are conserved densities for equations of any order.

Higher Order Conserved Densities (UP TO TOTAL DERIVATIVES):

$$\rho^{(1)} = a^{-1}(Da)^2 - \frac{12}{m(m+1)} Da\alpha_{(1)} + a\left[\frac{12}{m^2(m+1)}\alpha_{(1)}^2 - \frac{24}{m(m^2-1)}\alpha_{(2)}\right],$$

$$\begin{aligned}\rho^{(2)} &= a(Da)\left[D\alpha_{(1)} + \frac{3}{m}\alpha_{(1)}^2 - \frac{6}{(m-1)}\alpha_{(2)}\right] \\ &+ 2a^2\left[-\frac{1}{m^2}\alpha_{(1)}^3 + \frac{3}{m(m-1)}\alpha_{(1)}\alpha_{(2)} - \frac{3}{(m-1)(m-2)}\alpha_{(3)}\right],\end{aligned}$$

$$
\begin{aligned}
\rho^{(3)} &= a(D^2a)^2 - \frac{60}{m(m+1)(m+3)} a^2 D^2 a D\alpha_{(1)} + \frac{1}{4} a^{-1}(Da)^4 \\
&+ 30a(Da)^2 \left[\frac{(m-1)}{m(m+1)(m+3)} D\alpha_{(1)} + \frac{1}{m^2(m+1)} \alpha_{(1)}^2 - \frac{2}{m(m^2-1)} \alpha_{(2)} \right] \\
&+ \frac{120}{m(m^2-1)(m+3)} a^2 Da \left[-\frac{(m-1)(m-3)}{m} \alpha_{(1)} D\alpha_{(1)} + (m-3) D\alpha_{(2)} \right. \\
&- \left. \frac{(m-1)(2m-3)}{m^2} \alpha_{(1)}^3 + \frac{6(m-2)}{m} \alpha_{(1)}\alpha_{(2)} - 6\alpha_{(3)} \right] \\
&+ \frac{60}{m(m^2-1)(m+3)} a^3 \left[\frac{(m-1)}{m} (D\alpha_{(1)})^2 - \frac{4}{m} D\alpha_{(1)}\alpha_{(2)} + \frac{(m-1)(2m-3)}{m^3} \alpha_{(1)}^4 \right. \\
&- \left. 4\frac{(2m-3)}{m^2} \alpha_{(1)}^2 \alpha_{(2)} + \frac{8}{m} \alpha_{(1)}\alpha_{(3)} + \frac{4}{m} \alpha_{(2)}^2 - \frac{8}{(m-3)} \alpha_{(4)} \right].
\end{aligned}
$$

We compute $D_t\rho$, integrate by parts, until we obtain a term which is nonlinear it its highest derivative: The coefficient of this term should be zero. This gives partial differential equations that determine F.

First Result: Quasilinearity

Theorem *Let* $u_t = F[u]$ *be a scalar evolution equation of* $m = 2k+1$ *where* $k \geq 3$*, admitting a nontrivial conserved density*

$$\rho = Pu_n^2 + Qu_n + R$$

of order $n = m+1$*, where* P*,* Q *and* R *are independent of* u_m*. Then*

$$u_t = Au_m + B,$$

where A *and* B *are independent of* u_m*. [Bilge, 2005]*

The canonical density $\rho^{(1)}$ is of the form above, hence evolution equations of order greater than 5 are quasi-linear.

Second Result: Polynomiality in top 3 derivatives

Theorem *Let* $u_t = F[u]$ *be a scalar evolution equation of* $m = 2k+1$ *where* $k \geq 3$*, admitting the canonical conserved densities* $\rho^{(i)}$*,* $i = 1, 2, 3$*. Then*

$$u_t = \left(Au_m + Bu_{m-1}u_{m-2} + Cu_{m-2}^3\right) + \left(Eu_{m-1} + Gu_{m-2}^2\right) + (Hu_{m-2}) + (K)$$

where A*,* B*, ...,* K *depend on* x*,* t*,* $u, \ldots, u_{m-3}$*.*

[Bilge, Mizrahi 2008]

Third Result: 5th order quasilinear equations with non-constant separant:

Theorem *Let* $u_t = F[u]$, *where* F *depends on* x, t, $u, \ldots, u_5$ *and assume that* F *is quasilinear. Then*

$$u_t = a^5 u_5 + B u_4^2 + C u_4 + G$$

- *If all conserved are nontrivial,*

$$a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2}$$

- *If* ρ_3 *is trivial*

$$a = (\lambda u_3 + \mu)^{-1/3}$$

where α, β, γ, λ, μ *are functions of* x, t, u, u_1, u_2.

Forth Result: "Level homogeneous" equations of order 7, 9, 11 admidding "level homogeneous" conserved densities

- They are polynomial in u_k for $k \geq 4$

- In fact, it can be proved that if the equation is "level homogeneous", then the conserved densities are also level homogeneous.

- It needs to be proved that if it is polynomial and level homogeneous with respect to u_k, and the separant is independent of u_{k-1}, the equation will be polynomial and level homogeneous with respect to u_{k-1} (MAIN STEP)

- It has been shown that

$$u_t = (\alpha u_3^2 + \beta u_3 + \gamma)^{-m/2} u_m + B u_4 u_{m-1} + \ldots$$

  That is the separant is the same as for 5th order equations

Remarks

- Essentially non-linear classes of integrable equations arising at the third order are absent in higher orders!

- Although dependencies in all variables were used in the derivations, the equations relevant for obtaining polynomiality results involved the (non-polynomial) dependencies on top derivatives (First on u_m only, then on u_{m-1} only, and on u_{m-2} only at the last step).

- This observation led to the definition of a new type of grading on differential polynomials that we called "level above k";

Why 5th order is an exception to quasilinearity?

We start with $u_t = F$, use the existence of a conserved density $\rho^{(1)} \sim P u_{m+1}^2$, $m = 2k+1$ and $l = 1$ to get the homogeneous linear system

$$\begin{pmatrix} u_0(k,l) & u_1(k,l) \\ A_0(k,l) & D_0(k) + K_0(k) \end{pmatrix} \begin{pmatrix} PF'' \\ P'F' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The coefficient matrix is nonsingular for $k \neq 2$, that is except for 5th order equations. The term P is in fact nonzero for the first canonical density, hence $F'' = 0$. This proves quasilinearity. Here prime is the derivative with respect to top derivative u_{m-1}. Hence

$$u_t = A(x,t,u,\ldots,u_{m-1})u_m + B(x,t,u,\ldots,u_{m-1})$$

The final result is:

$$u_t = Au_m + Bu_{m-1}u_{m-2} + Cu_{m-2}^3 + Eu_{m-1} + Gu_{m-2}^2 + Hu_{m-2} + K$$

where the coefficients depend on at most u_{m-3}.

This suggests a new type of scaling...

The grading by "Level above k" Set up: Functions involved are polynomial in $u_{k+1}, \ldots$ but have arbitrary functional dependency on the variables x,t, $u, \ldots, u_k$.

Example: If $\varphi = \varphi(x, t, u, \ldots, u_k)$, then

$$D\varphi = \underbrace{\frac{\partial \varphi}{\partial u_k} u_{k+1}}_{\text{level 1}} + \alpha$$

$$D^2\varphi = \underbrace{\frac{\partial \varphi}{\partial u_k} u_{k+2} + \frac{\partial^2 \varphi}{\partial u_k^2} u_{k+1}^2}_{\text{level 2}} + \underbrace{\beta u_{k+1}}_{\text{level 1}} + \gamma,$$

$$D^3\varphi = \underbrace{\frac{\partial \varphi}{\partial u_k} u_{k+3} + 3\frac{\partial^2 \varphi}{\partial u_k^2} u_{k+1} u_{k+2} + \frac{\partial^3 \varphi}{\partial u_k^3} u_{k+1}^3}_{\text{level 3}} + \underbrace{\mu u_{k+2} + \nu u_{k+1}^2}_{\text{level 2}} + \underbrace{\lambda u_{k+1}}_{\text{level 1}} + \eta,$$

where α, β etc. depend on at most u_k.

We define the "level above k" as the total number of derivatives above k.

Examples of level homogenous equations:

Mikhailov-Shabat-Sokolov: Classification of fifth order equations

$$u_t = u_5 + f(x, t, u, u_1, u_2, u_3, u_4).$$

Up to dependencies on x, t, u, u_1, they obtain the form

$$u_t = u_5 + (A_1 u_2 + A_2) u_4 + A_3 u_3^2 + (A_4 u_2^2 + A_5 u_2 + A_6) u_3 + A_7 u_2^4 + A_8 u_2^3 + A_9 u_2^2 + A_a 10 u_2 + A_{11}.$$

Rearranging,

$$u_t = \underbrace{u_5 + A_1 u_2 u_4 + A_3 u_3^2 + A_4 u_2^2 u_3 + A_7 u_2^4}_{\text{level 4 above k=1}}$$

$$+ \underbrace{A_2 u_4 + A_5 u_2 u_3 + A_8 u_2^3}_{\text{level 3 above k=1}}$$

$$+ \underbrace{A_6 u_3 + A_9 u_2^2}_{\text{level 2}} + \underbrace{A_{10} u_2}_{\text{level 1}} + \underbrace{A_{11}}_{\text{level 0}} .$$

At fifth order, level homogeneity stops here, one needs to use differential substitutions to simplify and recover level homogeneous expressions.

Why we use this grading?

Level above k is invariant under integration by parts.: Let $p_1 < p_2 < \cdots < p_l < s-1$. Then

$$\begin{aligned} \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s &\cong -D\left(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\right) u_{s-1}, \\ \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_{s-1}^{p} u_s &\cong -\frac{1}{p+1} D\left(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\right) u_{s-1}^{p+1}. \end{aligned}$$

The integration by parts is repeated until one encounter a non-integrable monomial as:

$$u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s^{p}, \quad p > 1.$$

In symbolic computation: Top level terms depend on the top derivative only. Use only the dependency on the top derivative.

In determining the form of canonical densities: If the evolution equation is level homogeneous above k, its recursion operator is also level homogeneous. If the canonical densities are nontrivial, there are level homogeneous conserved densities (proved).

If F is level homogeneous above every base level, how do we write F?

This uses the "decomposition of integers"

Base u_k							
u_{m-1}	1						
u_{m-2}	2	1+1					
u_{m-3}	3	2+1	1+1+1				
u_{m-4}	4	3+1 2+2	2+1+1	1+1+1+1			
u_{m-5}	5	4+1 3+2	3+1+1 2+2+1	2+1+1+1	1+1+1+1+1		
u_{m-6}	6	5+1 4+2 3+3	4+1+1 3+2+1 2+2+2	3+1+1+1 2+2+1+1	2+1+1+1+1	6×1	
u_{m-7}	7	6+1 5+2 4+3	5+1+1 4+2+1 3+3+1 3+2+2	4+1+1+1 3+2+1+1 2+2+2+1	3+1+1+1+1 2+2+1+1+1	2+1+1+1+1+1	7×1

7th order equations: Top level terms above each level.

Base	Level							
u_6	1	u_7 (1)						
u_5	2	u_7 (2)	u_6^2 (1+1)					
u_4	3	u_7 (3)	u_6u_5 (2+1)	u_5^3 (1+1+1)				
u_3	4	u_7 (4) u_5u_5 (2+2)	u_6u_4 (3+1)	$u_5u_4u_4$ (2+1+1)	u_4^4 (4×1)			
u_2	5	u_7 (5) u_5u_4 (3+2)	u_6u_3 (4+1) $u_4u_4u_3$ (2+1+1)	$u_5u_3u_3$ (3+1+1)	$u_4u_3^3$ $(2+3\times 1)$	u_3^5 (5×1)		
u_1	6	u_7 (6) u_5u_3 (4+2) u_3u_3 (3+3)	u_6u_2 (5+1) $u_4u_3u_2$ (3+2+1) $u_2u_2u_2$ (2+2+2)	$u_5u_2u_2$ (4+1+1) $u_3^2u_2^2$ (2+2+1+1)	$u_4u_2^3$ $(3+3\times 1)$	$u_3u_2^4$ $(2+4\times 1)$	u_2^6 6×1	
u	7	u_7 u_5u_2 u_4u_3	u_6u_1 $u_4u_2u_1$ $u_3u_3u_1$ $u_3u_2u_2$	$u_5u_1^2$ $u_3u_2u_1^2$ $u_2^3u_1$	$u_4u_1^3$ $u_2^2u_1^3$	$u_3u_1^4$	$u_2u_1^5$	u_1^7

How can we use level homogeneity?

- We have proved level homogeneity in top three derivatives,

- We have proved that if the equation is level homogeneous and if it admits a recursion operator, the operator is also level homogeneous, provided that the separant is of level zero (this involves solutions of first order ODE's).

- It follows that the canonical densities are level homogeneous.

- The canonical densities may or may not be trivial. If they are trivial, this gives PDE's for F, in $u_t = F$. For example, the triviality of ρ_3 leads to the Sawada-Kotera, Kaup equations.

- If all conserved densities are nontrivial we expect to obtain hierarchies related to the KdV.

- MAIN PROBLEM: Prove in general that level homogeneity above a base k and the existence of conserved densities lead to level homogeneity above the base level $k-1$. This involves non-singularity of systems of algebraic equations at the top level,

- But for lower levels, we have to solve differential equations; their solutions may a priori involve logartithms and algebraic functions, but the coefficients of such terms vanish (observed at order 5).

- The whole scheme breaks down when $\partial a/\partial u_k$ is nonzero. This is the case $k = 3$

- After this stage we have arbitrary functions, we should find transformations to eliminate them.

How to explain missing conserved densities?

- The Korteweg-deVries hierarchy have a recursion operator of order 2 and has a starting symmetry u_x, hence there are symmetries, conserved covariants (co-symmetries) and conserved densities at every other order. Another interpretation is that the Lax operator is of order 2 and even order symmetries are missing.

- The Sawada-Kotera and Kaup hierarchies have recursion operators of order 6 with two starting symmetries of orders 1 and 5, hence it has symmetries at orders $1 + 6k$ and $5 + 6k$. An alternative interpretation is that the Lax operator has order 3 and symmetries are missing at orders that are multiples of 3.

Can we use transformations involving higher order derivatives?

Given $u_t = F(x, t, u, \ldots, u_m)$, can we set

$$\frac{\partial F}{\partial u_m} = 1, \quad \text{and} \quad \frac{\partial F}{\partial u_{m-1}} = 0?$$

If $D_t\rho = D_x\eta$ and ρ is nontrivial, then $dx' = \rho(u, u_1)dx + \eta(u, u_1, \ldots, u_n)dt$, $t' = t$, $u' = \psi(u)$, $u'_k = \left(\frac{1}{\rho}D\right)^k \psi$, $k = 1, 2, \ldots$. [MSS, p.127] defines a locally an invertible transformation.

- Consider $u_t = A(u, u_1)u_m + B(u, u_1, \ldots, u_{m-1})$,
- $\rho_{-1} = A^{-1/m}$ is always a conserved density,
- Use this to set $A = 1$.

If A depends on higher derivatives, the transformed equation may not be local! (This is our main problem now)

What's next?

- Scale invariance/level homogeneity is a consequence of the existence of the recursion operator,

- Can we set $A = 1$ by a differential transformation and show that integrability guarantees that the resulting equation is local?

- The form of the exponents $-1/2$ and $-1/3$ in the separant a suggests that they are related to 2nd and 3rd order Lax operators.

Thank you...

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