VARIATIONS ON A THEOREM OF KOROVKIN

Contents

1 Introduction 2

2 Preliminaries and Notations 3

3 Approximation Theorems 21

4 Variations on a Theorem of Korovkin 27

5 Applications to Classical Results 32
1 Introduction

The basic problem is to approximate functions with simpler functions. This is a subject of *Approximation Theory*. One of the most impressive results in the theory is the so-called Bohman-Korovkin Theorem on convergence of positive linear operators on a space of continuous functions. This theorem is widely known as Korovkin Theorem. It arises from a generalization of the proof of the Weierstrass Approximation Theorem. The simplicity of this approximation theorem yield not only a wide range of applications but also a whole new theory: the so-called Korovkin-type Approximation Theory. For a detailed survey you may see [1].

In 1953, for a sequence of positive linear operators \( \{L_n\}_{n \geq 1} \) defined on the space \( C[a,b] \) with the supremum norm, Korovkin discovered the following simple criterion:

**Theorem of Korovkin**: \( \{L_n(f)\}_{n \geq 1} \) converges uniformly to \( f \) on \( [a,b] \), for each \( f \in C[a,b] \), if it converges uniformly to \( f \) on \( [a,b] \), only for \( f \in \{1, x, x^2\} \).

We will discuss this approximation in some later time in this project.

There are five sections in this project.

In the second section, the fundamental content of this project will be given.

In the third section, some relevant approximation theorems together with the Korovkin theorem will be proven clasically.

In the forth section, the definition of bounding functions and some lemmas which will be used to give a new proof for Korovkin Theorem will be given.

In the last section, with the aid of these functions, lemmas and suitable sequences of positive linear operators (defined on the space of continuous real-valued functions), several well-known approximation theorems will be proven again.

We have two main references. One of them is a book written by G.G.Lorentz[2] and the other one is an article written by Hector E. Lomeli and Cesar Garcia[3].
2 Preliminaries and Notations

First we start with compact spaces.

Definition 2.1

(i) A collection of open sets \( \{ U_i : i \in I \} \) in \( X \) is an open cover of \( Y \subset X \), if \( Y \subset \bigcup_{i \in I} U_i \).

(ii) A subcover of \( \{ U_i : i \in I \} \) is a subcollection \( \{ U_j : j \in J \} \) for some \( J \subset I \) that still covers \( Y \). \( \{ U_j : j \in J \} \) is a finite subcover if \( J \) is finite.

(iii) We say that a space \( X \) is compact, if every open cover of \( X \) has a finite subcover.

Definition 2.2

(i) A metric on a space \( X \) is a function

\[
d : X \times X \mapsto \mathbb{R}^+
\]

\[
(x, y) \mapsto d(x, y)
\]

with the properties:

1. \( d(x, y) = 0 \iff x = y \)
2. \( d(x, y) = d(y, x) \)
3. \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

(ii) The pair \((X, d)\) is called a metric space.

Next definition is about vector spaces(see[4]).
Definition 2.3 (Vector Spaces/ Linear Spaces)

We say that a set $U$, with addition and scalar multiplication, is a vector space over the field $F$, if it satisfies the followings:

- $\forall u, v \in U$ $u + v \in U$.
  - (i) $\forall u, v \in U$ $u + v = v + u$.
  - (ii) $\forall u, v, w \in U$ $u + (v + w) = (u + v) + w$.
  - (iii) $\exists 0 \forall u \in U$ such that $u + 0 = u$.
  - (iv) $\forall u \in U$ $\exists v \in U$ such that $u + v = 0$.

- $\forall u \in U \forall a \in F$ $au \in U$.
  - (v) $\forall u \in U \forall a, b \in F$ $a(bu) = (ab)u$.
  - (vi) $\forall u, v \in U \forall a \in F$ $a(u + v) = au + av$.
  - (vii) $\forall u \in U \forall a, b \in F$ $(a + b)u = au + bu$.
  - (viii) $\exists 1_F \forall u \in U$ $1_F u = u$.

Let us now give a significant example of a linear space.

Example 2.1 (the space of real functions)

Let $X$ be any non-empty set. Let $F(X)$ be the set of all real-valued functions:

$F(X) = \{ f : X \rightarrow \mathbb{R} \}$

The sum of the functions is defined by

$(f + g)(x) := f(x) + g(x)$ $\forall x \in X$, $\forall f, g \in F(X)$, (2.1)

and the multiplication of the function with a scalar $\alpha$ is defined by

$(\alpha f)(x) := \alpha f(x)$ $\forall x \in X$, $\forall f \in F(X)$. (2.2)

Now we will show that $F(X)$ is a linear space.

(i) $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ $\forall x \in X$, $f + g = g + f$ $\forall f, g \in F(X)$.  

4
(ii) 
\[ f + (g + h)(x) = (f + g)(x) + h(x) = (f + g + h)(x) \quad \forall x \in X, \]
\[ f + (g + h) = (f + g) + h \quad \forall x \in X, \quad \forall f, g, h, \in \mathcal{F}(X). \]

(iii) \( \exists \Phi \in \mathcal{F}(X) \) \( \forall f \in \mathcal{F}(X) \) such that \( f + \Phi = f \) where \( \Phi(x) = 0 \) \( \forall x \in X \), since
\[ (f + \Phi)(x) = f(x) + \Phi(x) = f(x) + 0 = f(x). \]

(iv) \( \forall f \in \mathcal{F}(X) \) define a function \( -f \) such that \( (-f)(x) = -f(x) \).

Claim: \( f + (-f) = \Phi \).
\[ [f + (-f)](x) = f(x) + (-f)(x) = f(x) - f(x) = 0 \quad \forall x \in X, \]
as we desired.

(v) \( (1f)(x) = 1f(x) = f(x) \) implies \( 1f = f \) where \( 1 \) is the identity element in \( \mathbb{R} \).

(vi) \( \forall \alpha, \beta \in \mathbb{R}, f \in \mathcal{F}(X) \) \( (\alpha \beta) f = \alpha (\beta f) \) since
\[ [(\alpha \beta) f](x) = (\alpha \beta)(f)(x) = \alpha (\beta f(x)) = [\alpha (\beta f)](x) \quad \forall x \in X. \]

(vii) \( [\alpha (f + g)](x) = \alpha [f(x) + g(x)] = \alpha f(x) + \alpha g(x) = [\alpha f + \alpha g](x) \quad \forall x \in X \]
\[ \forall f, g \in \mathcal{F}(X), \forall \alpha \in \mathbb{R}. \]

(viii) \( \forall \alpha, \beta \in \mathbb{R}, f \in \mathcal{F}(X) (\alpha + \beta) f = \alpha f + \beta f \) since
\[ [(\alpha + \beta) f](x) = (\alpha + \beta) f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x) \quad \forall x \in X. \]

Hence \( \mathcal{F}(X) \) is a linear space over the reals.

**Definition 2.4** A subset \( W \) of a vector space \( V \) over a field \( \mathbb{F} \) is called a **subspace** of the vector space \( V \) if:

1. \( W \neq \emptyset \).
2. \( W \) is closed under addition,
3. \( W \) is closed under scalar multiplication.

**Definition 2.5** We say that a function \( f \) is **continuous** on \( X \) if:
\[ \forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \ that \quad |f(x) - f(a)| < \epsilon \quad when \quad |x - a| < \delta \quad \forall x \in X. \]
Example 2.2 (the space of continuous real-valued functions)

We claim that $C(X)$, the set of continuous real-valued functions, is a subspace of $F(X)$.

1. $C(X) \neq \emptyset$ since $\Phi(x) = 0$ is continuous.

2. Assume $f$ and $g$ are continuous on $X$. Then

   $\forall \epsilon > 0 \exists \delta_1(\epsilon) > 0$ such that $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta_1 \forall a \in X$.

   $\forall \epsilon > 0 \exists \delta_2(\epsilon) > 0$ such that $|g(x) - g(a)| < \epsilon$ when $|x - a| < \delta_2 \forall a \in X$.

   Choose $\delta = \min\{\delta_1, \delta_2\}$. Then

   $$|(f + g)(x) - (f + g)(a)| = |f(x) + g(x) - f(a) - g(a)|$$

   $$\leq |f(x) - f(a)| + |g(x) - g(a)| < 2\epsilon$$

   when $|x - a| < \delta$. Hence we obtain

   $$f + g \in C(X).$$

3. Assume $f$ is continuous. Then $\forall a \in X, \forall \beta \in \mathbb{R}$, we have

   $$|(\beta f)(x) - (\beta f)(a)| = |\beta f(x) - \beta f(a)| = |\beta||f(x) - f(a)|.$$  

   But $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta$. Hence

   $$|(\beta f)(x) - (\beta f)(a)| < \epsilon|\beta|$$

   which means that $\beta f \in C(X)$.

Therefore the set $C(X)$ is also a space over the reals.

Definition 2.6

Let $v \in V$. The function $\|\cdot\| : V \to \mathbb{R}$ is defined by:

(i) $\|v\| \geq 0$,

(ii) $\|v\| = 0$ if and only if $v = 0$,

(iii) $\|\alpha v\| = |\alpha|\|v\|$,

(iv) $\|u + v\| \leq \|u\| + \|v\|$ (Triangular Inequality).
Then $\|v\|$ is called the **norm** of the element $v$. In this case, $(V, \|\|)$ is called a **normed** space.

**Example 2.3 (supremum norm)**

We claim that the function $\|f\| = \sup_{x \in X} |f(x)|$ is a norm on $\mathcal{C}(X)$.

(i) $\|f\| = \sup_{x \in X} |f(x)| \geq 0$, $\forall f \in \mathcal{C}(X)$.

(ii) $\|f\| = 0$ implies $\sup_{x \in X} |f(x)| = 0$. Hence $f(x) = 0$, $\forall x \in X$. So we obtain $f = \Phi$.

On the other hand, if $f = \Phi$, then $f(x) = \Phi(x)$, $\forall x \in X$. Hence $\sup_{x \in X} |f(x)| = \|f\| = 0$.

(iii) $\|(\alpha f)\| = \sup_{x \in X} |(\alpha f)(x)| = \sup_{x \in X} |\alpha f(x)| = \sup_{x \in X} |\alpha| |f(x)| = |\alpha| \|f\|$. Hence $\|(\alpha f)\| = |\alpha| \|f\|$.

(iv) $\|f + g\| = \sup_{x \in X} |(f + g)(x)| = \sup_{x \in X} |f(x) + g(x)| \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| = \|f\| + \|g\|$. So we obtain $\|f + g\| \leq \|f\| + \|g\|$.

**Lemma 2.1**

Let $\{f_n\}$ be a sequence of real-valued functions defined on $X$. Then

$f_n \to f$ uniformly if and only if $\|f_n - f\| \to 0$.

*Proof.* Assume $f_n$ converges uniformly to $f$ on $X$. Then

$$\forall \epsilon > 0 \ \exists N(\epsilon) \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \epsilon \text{ when } n > N \ \forall x \in X,$$

$$\sup_{x \in X} |f_n(x) - f(x)| = \sup_{x \in X} |(f_n - f)(x)| = \|f_n - f\| < \epsilon,$$

$$\|(f_n - f)\| \to 0.$$ On the other hand, assume $\|f_n - f\| \to 0$. Then
∀\(\epsilon > 0\) \(\exists N(\epsilon) \in \mathbb{N}\) such that \(\|f_n - f\| < \epsilon\) when \(n > N\),

\[
\|f_n - f\| = \sup_{x \in X}|(f_n - f)(x)| = \sup_{x \in X}|f_n(x) - f(x)| < \epsilon \text{ implies } |f_n(x) - f(x)| < \epsilon,
\]

\(f_n \to f\) uniformly.

\textbf{Corollary 2.1}

The convergence \(f_n \to f\) in the norm of \(C(X)\), that is \(\|f_n - f\| \to 0\) as \(n \to \infty\), is equivalent to the uniform convergence of \(f_n(x)\) to \(f(x)\) for all \(x \in X\).

It follows from this interpretation that the space \(C(X)\) is complete: if \(f_n\) is a Cauchy sequence, (that is, \(\|f_n - f_m\| \to 0\) for \(n, m \to \infty\)) then \(f_n\) converges to some element \(f\) of \(C(X)\): \(\|f_n - f\| \to 0\).

\textbf{Definition 2.7 (Complete Normed Spaces/Banach Spaces)}

A normed space is complete, if every Cauchy sequence is convergent. A complete normed space is called a \textbf{Banach Space}.

Now, we will consider two examples. Our aim is to show that completeness depends on the norm defined on the space we have chosen.

\textbf{Example 2.4}

The space of continuous real-valued functions \(C(X)\) with the supremum norm

\[
\|f\| = \sup_{x \in X}|f(x)|,
\]

is a complete normed space, that is a Banach space by Corollary (2.1).

Now, we will give another example of norm defined on \(C(X)\).

\textbf{Example 2.5} Let \(\|f\|_1 = \int_0^1 |f(x)|dx\).

We can easily show that \(\|\cdot\|_1\) is a norm. Clearly

(i) \(\|f\|_1 = \int_0^1 |f(x)|dx \geq 0\).
(ii) \( \|f\|_1 = 0 \) implies \(|f(x)| = 0 \) Hence \( f = 0 \).

On the other hand, if \( f = 0 \), then \( \int_0^1 |f(x)|dx = \|f\|_1 = 0 \).

(iii) \( \|\alpha f\|_1 = \int_0^1 |(\alpha f)(x)|dx = \int_0^1 |\alpha||f(x)|dx = |\alpha||f\|_1 \).

(iv) \( \|f + g\|_1 = \int_0^1 |(f + g)(x)|dx \leq \int_0^1 |f(x)| + |g(x)|dx = \|f\|_1 + \|g\|_1 \).

Let us take a sequence

\[
h_n(x) = \begin{cases} 
0 & 0 \leq x \leq \frac{1}{2} \\
n(x - \frac{1}{2}) & \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\
1 & \frac{1}{2} + \frac{1}{n} < x \leq 1 
\end{cases}
\]

Observe that \( h_n \) is continuous.

We can show that \( \{h_n\}_{n \geq 1} \) is a Cauchy sequence. Without loss of generality assume \( n > m \):

\[
\|h_n - h_m\|_1 = \int_0^1 |h_n(x) - h_m(x)|dx \\
= \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} \left[ n(x - \frac{1}{2}) - m(x - \frac{1}{2}) \right] dx + \int_{\frac{1}{2} + \frac{1}{n}}^{\frac{1}{2} + \frac{1}{m}} \left[ 1 - m(x - \frac{1}{2}) \right] dx \\
= \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} \left[ (n - m)(x - \frac{1}{2}) \right] dx + \int_{\frac{1}{2} + \frac{1}{n}}^{\frac{1}{2} + \frac{1}{m}} \left[ 1 - mx + \frac{m}{2} \right] dx \\
= \frac{n - m}{2} \left[ \frac{1}{2} + \frac{1}{n} \right] - \frac{1}{2m} \left[ (1 - mx + \frac{m}{2}) \right]_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{m}} \\
= \frac{n - m}{2n^2} - \frac{(n - m)^2}{2m^2n^2} \to 0 \text{ as } n, m \to \infty.
\]

Suppose \( h_n \) tends to \( h \) with respect to the norm \( \|\cdot\|_1 \). So we have:

\[
f_0^\frac{1}{2} |h(x)|dx = f_0^\frac{1}{2} |h(x) - h_n(x)|dx \leq f_0^1 |h(x) - h_n(x)|dx = \|h - h_n\|_1 \to 0 \text{ implies } h(x) \to 0.
\]

Similarly \( \int_{\frac{1}{2} + \epsilon}^1 |h(x) - 1|dx \leq \|h - h_n\|_1 \to 0 \) implies \( h(x) = 1 \) on \([1/2 + \epsilon, 1]\) where \( 0 < \epsilon < 1/2 \).

Therefore \( h(x) \) is not continuous. Hence \( C(X) \) is not a Banach space with respect to the norm \( \|\cdot\|_1 \).
Definition 2.8 (subalgebra)

We say that a linear subspace $E$ of $\mathcal{F}(X)$ is a subalgebra if

$$f, g \in E \text{ for every } f, g \in E.$$ 

Definition 2.9 (degree of a trigonometric polynomial)

$$T_n(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

is a trigonometric polynomial of degree $n$.

Definition 2.10 (Linear Operators)

The operator $L : X \mapsto Y$, where $X$ and $Y$ are Banach spaces, is said to be linear if it satisfies:

(i) $L(f + g) = L(f) + L(g)$

(ii) $L(\alpha f) = \alpha L(f) \quad \forall f, g \in X, \quad \alpha \in \mathbb{R}$

In particular if we choose $Y = \mathbb{R}$, then $L$ is called linear functional.

Let us consider another property of linear operators.

Definition 2.11 We say that a linear operator is bounded if for some constant $M > 0$,

$$\|L(f)\| \leq M\|f\|, \quad f \in X.$$ 

We can also define the norm of $L$, $\|L\|$ as:

$$\|L\| = \inf \{M : \|L(f)\| \leq M\|f\|\}.$$ 

$$\|L\| = \sup_{f \neq 0} \left(\frac{L(f)}{\|f\|}\right) = \sup_{f \neq 0} \left[\frac{L(f)}{\|f\|}\right] = \sup_{\|f\|=1} \|L(f)\|.$$ 

Lemma 2.2 A bounded linear operator is continuous.

Proof. $\forall f, g \in C(X), \quad \|L(f) - L(g)\| = \|L(f - g)\| \leq \|L\|\|f - g\| < \|L\| \delta$, when $\|f - g\| < \delta$. Hence $L$ is continuous. ■

Recall that we say a real function $f$ is positive if $\forall x \in X, \quad f(x) \geq 0$ where $f \in C(X)$, namely $f \geq 0$.

Now we are ready to give the definition of a positive linear operator:
Definition 2.12 (Positive Linear Operators) An operator \( L : C(X) \to C(X) \) is called **positive** if it maps each positive function to a positive function.

Now let us consider some properties of such operators defined on a set of continuous real-valued functions:

**Properties of Positive Linear Operators**

(i) Let \( f, g \) be members of \( C(X) \). If \( f \leq g \), then \( L(f) \leq L(g) \).

Proof. If \( g - f \geq 0 \), then \( L(g - f) = L(g) - L(f) \geq 0 \). Hence \( L(g) \geq L(f) \) ■

(ii) If \( L \) is positive and \( f \in C(X) \), then we have

\[
|L(f)(x)| \leq L(f)(x) \quad \forall x \in X \tag{2.3}
\]

Proof. Since \( f(x) \leq |f(x)| \), we have \(-|f(x)| \leq f(x) \leq |f(x)|\). By using the second part of the inequality we obtain :

\[
L|f(x)| \geq L(f(x)) \tag{2.4}
\]

On the other hand, if we use the first part of the inequality \(-|f(x)| \leq f(x) \leq |f(x)|\), we get \( |f(x)| \geq -f(x) \), and hence

\[
L|f(x)| \geq L(-f(x)) = -L(f(x)) \tag{2.5}
\]

By (2.4) and (2.5) we have \( L|f(x)| \geq |Lf(x)| \) ■

We can also observe that

\[
|L(f)| \leq L(\|f\|) \leq L(\|f\|e) \leq \|f\|L(e), \text{ where } e(x) = 1,
\]

\[
|L(f)| \leq \|f\|L(e),
\]

\[
\sup \frac{|L(f)|}{\|f\|} \leq \sup |L(e)| = \|L(e)\|,
\]

\[
\|L\| \leq \|L(e)\|. \tag{2.6}
\]

On the other hand, we have

\[
|L(f)| \leq \|f\|L(e),
\]

\[
\sup |L(f)\| \leq \sup \frac{|L(e)|}{\|e\|} = \|L\|,
\]

\[
\sup |L(e)| \leq \sup \frac{|L(e)|}{\|e\|} = \|L\|. \tag{2.7}
\]
By (2.6) and (2.7), we have \( \|L\| = \|L(e)\| \).

Let us give some examples of positive linear operators.

**Example 2.6 (Bernstein Polynomials)**

For a function \( f \) defined on \([0, 1]\), the Bernstein Polynomial is defined by:

\[
B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k}{n} \right), \quad n = 0, 1, \cdots.
\]  

For example if we let \( n = 2 \) the polynomial becomes:

\[
B_2(f, x) = \sum_{k=0}^{2} \binom{2}{k} x^k (1 - x)^{2-k} f \left( \frac{k}{2} \right) = \binom{2}{0} x^0 (1 - x)^2 f(0) + \binom{2}{1} x^1 (1 - x)^1 f \left( \frac{1}{2} \right) + \binom{2}{2} x^2 (1 - x)^0 f(1)
\]

\[
= (1 - x)^2 f(0) + 2x(1 - x)f \left( \frac{1}{2} \right) + x^2 f(1)
\]

Note that

\[
B_n(e, x) = \sum_{k=0}^{n} p_{nk}(x) = 1, \quad \text{where } p_{nk}(x) = \binom{n}{k} x^k (1 - x)^{n-k} \geq 0
\]  

We can easily show that the polynomial is linear. Assume we have \( f, g \) defined on \([0, 1]\).

\[
B_n(f + g)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} (f + g) \left( \frac{k}{n} \right) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k}{n} \right) + \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} g \left( \frac{k}{n} \right) = B_n(f, x) + B_n(g, x)
\]

Similarly we have

\[
B_n(\alpha f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} (\alpha f) \left( \frac{k}{n} \right) = \alpha \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k}{n} \right) = \alpha B_n(f, x)
\]
For every \( f \geq 0 \) all terms become positive and summation of positive terms is again positive. Hence the polynomial (2.8) is positive.

In sum, the Bernstein polynomial of a function defined on the interval \([0, 1]\) is, in fact, a positive linear operator.

Now we will try to find the norm of this operator:

\[
B_n(e, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = 1
\]  

(2.10)

\[
\frac{|B_n(f, x)|}{\|f\|} = \frac{B_n \left( \frac{f}{\|f\|}, x \right)}{|B_n(e, x)|} = \frac{|B_n(e, x)|}{\|e\|},
\]

\[
\sup \frac{|B_n(f, x)|}{\|f\|} = \frac{|B_n(e, x)|}{\|e\|} = 1,
\]

and hence we have \( \|B_n\| = \|B_n(e)\| = 1 \).

**Example 2.7 (Fourier Series)**

Let \( f \) be a \( 2\pi \)-periodic, integrable function. The coefficients of its Fourier Series,

\[
a_0 = \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right)
\]

(2.11)

are given by the formulas

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt.
\]

(2.12)

We consider the \( n \)th partial sum \( s_n \) of the series (2.11). We can find this sum in the following way. Write \( s_n = u_0 + u_1 + \cdots + u_n \) where

\[
u_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} f(t) \, dt,
\]

\[
u_k = a_k \cos kx + b_k \sin kx \]

\[
= \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt \right] \cos kx + \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt \right] \sin kx
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \cos kt \cos kx + \sin kt \sin kx \right] \, dt
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos (k(t - x)) \, dt. \quad k = 1, 2, \ldots.
\]
So we have:

\[ s_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt + \sum_{k=1}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos k(t-x) \, dt \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2} + \cos(t-x) + \cdots + \cos n(t-x) \right] \, dt. \]

Hence

\[ s_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t-x) \, dt, \quad (2.13) \]

where

\[ D_n(t-x) = \frac{1}{2} + \cos(t-x) + \cdots + \cos n(t-x). \]

The function \( D_n \) is called the \( n \)-th Dirichlet’s kernel (see [5]). Let \( t - x = \alpha \), then

\[ D_n\alpha = \frac{1}{2} + \cos \alpha + \cdots + \cos n\alpha \]

\[ (2.14) \]

If we multiply and divide the equation (2.14) by \( 2 \sin \frac{\alpha}{2} \) at the same time we obtain:

\[ D_n\alpha = \frac{\sin \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \cos \alpha + \cdots + 2 \sin \frac{\alpha}{2} \cos n\alpha}{2 \sin \frac{\alpha}{2}}. \]

But we know that

\[ 2 \sin \frac{\alpha}{2} \cos \alpha = \sin \frac{3\alpha}{2} - \sin \frac{\alpha}{2}, \]

\[ 2 \sin \frac{\alpha}{2} \cos 2\alpha = \sin \frac{5\alpha}{2} - \sin \frac{3\alpha}{2}, \]

\[ \vdots \]

\[ 2 \sin \frac{\alpha}{2} \cos n\alpha = \sin \frac{(2n+1)\alpha}{2} - \sin \frac{(2n-1)\alpha}{2}. \]

Thus, after some cancellations we have the following:

\[ D_n\alpha = \frac{\sin \frac{(2n+1)\alpha}{2}}{2 \sin \frac{\alpha}{2}}. \]

So we obtain the partial sum \( s_n \) as:

\[ s_n = s_n(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin \left[ \frac{(2n+1)(t-x)}{2} \right] \frac{2 \sin \left( \frac{t-x}{2} \right)}{2 \sin \frac{\alpha}{2}} \, dt. \quad (2.15) \]

Now, we define \( \sigma_n \) as arithmetic mean (or Cesaro average) of \( s_n \):

\[ \sigma_n = \sigma_n(f, x) = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} \]

\[ = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{\sin \frac{\alpha}{2} + \sin \frac{3\alpha}{2} + \cdots + \sin \frac{(2n-1)\alpha}{2}}{2 \sin \frac{\alpha}{2}} \right] \, dt. \]
Similarly, if we multiply and divide the bracket in the integrant by $2 \sin \frac{\alpha}{2}$ we obtain:

$$T_n \alpha = \frac{2 \sin \frac{\alpha}{2} \sin \frac{\alpha}{2} + 2 \sin \frac{3\alpha}{2} \sin \frac{\alpha}{2} + \cdots + 2 \sin \frac{(2n-1)\alpha}{2} \sin \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2}}.$$ 

But we know that

$$2 \sin \frac{\alpha}{2} \sin \frac{\alpha}{2} = 1 - \cos \alpha,$$
$$2 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2} = \cos \alpha - \cos 2\alpha,$$
$$\vdots$$
$$2 \sin \frac{\alpha}{2} \sin \frac{(2n-1)\alpha}{2} = \cos(n-1)\alpha - \cos n\alpha.$$ 

After some calculations, we obtain the followings:

$$T_n \alpha = \frac{1 - \cos n\alpha}{2 \sin \frac{\alpha}{2}} = \frac{\sin^2 \frac{n\alpha}{2}}{\sin \frac{\alpha}{2}},$$
$$\sigma_n = \frac{1}{\pi n} \int_{-\pi}^{\pi} f(t) \frac{\sin^2 \frac{n\alpha}{2}}{2 \sin^2 \frac{\alpha}{2}} dt = \frac{1}{2\pi n} \int_{-\pi}^{\pi} f(t) \left[ \frac{\sin \frac{n(t-x)}{2}}{\sin \frac{t-x}{2}} \right]^2 dt. \quad (2.16)$$

For alternative calculations you may see [5].

Observe that $\{\sigma_n\}$ is a sequence of positive linear operators mapping continuous $2\pi$-periodic functions to $2\pi$-periodic functions.

Now, let us show that $\|\sigma_n\| = 1$. Since $\|\sigma_n\| = \|\sigma_n(e)\|$ where $e(x) = 1$.

$s_0 = u_0, \quad s_1 = u_0 + u_1, \cdots, s_{n-1} = u_0 + u_1 + \cdots + u_{n-1}$, with $f(x) = e(x)$.

$$u_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dt = 1$$
$$u_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t-x) dt$$
$$= \frac{1}{\pi} [\sin(\pi-x) - \sin(-\pi-x)]$$
$$= \frac{1}{\pi} [\sin x + \sin(\pi + x)] = 0$$

For arbitrary $k \geq 1$, we have

$$u_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos k(t-x) dt$$
$$= \frac{1}{k\pi} [\sin(k\pi - kx) - \sin(-k\pi - x)]$$
$$= \frac{1}{k\pi} [\sin(k\pi - kx) + \sin(k\pi + kx)]$$
$$= \frac{1}{k\pi} [(-1)^{k+1} \sin kx + (-1)^k \sin(kx)] = 0$$
Hence \( s_n(x) \equiv 1 \) for the function \( e(x) = 1 \). So we have the following:

\[
\|\sigma_n\| = \|\sigma_n(e)\| = \frac{1 + 1 + \cdots + 1}{n} = 1.
\] (2.17)

**Corollary 2.2**

The operators \( s_n \) are also linear, but not necessarily positive. It follows from the definition of \( s_n \) and \( \sigma_n \) that for each \( f \), both \( s_n(f, x) \) and \( \sigma_n(f, x) \) are trigonometric polynomials of degree \( n \) and \( n - 1 \), respectively.

Let \( K \) be an additive group of real numbers modulo \( 2\pi \). Let \( C^\ast = C[K] \) be a set of continuous \( 2\pi \)-periodic real-valued functions. If \( f \in C^\ast \), then the norm of \( s_n \) is given by the following theorem:

**Theorem 2.1 (Fejer)** The norm \( \|s_n\| \) of the operator \( s_n(f, x) \) is equal to

\[
A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((2n + 1)(t/2))}{2 \sin(t/2)} \right| \, dt = \frac{4}{\pi^2} \log n + O(1); \quad (2.18)
\]

also, the norm of \( s_n(f) \), for each fixed \( x \), considered as a linear functional from \( C^\ast \) to \( \mathbb{R} \), is equal to \( (2.18) \).

**Proof.** Let

\[
D_n(t - x) = \frac{\sin \left( \frac{(2n + 1)(t-x)}{2} \right)}{2 \sin \left( \frac{t-x}{2} \right)}.
\]

Consider the absolute value of \( s_n(f, x) \):

\[
|s_n(f, x)| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)D_n(t-x) \, dt \right| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \sup_{t \in [-\pi, \pi]} |f(t)||D_n(t-x)| \, dt
\]

\[
= \|f\| \int_{-\pi}^{\pi} |D_n(t)| \, dt = A_n \|f\|,
\]

\[
\sup \frac{|s_n(f, x)|}{\|f\|} \leq A_n,
\]

\[
\|s_n\| \leq A_n. \quad (2.19)
\]

This inequality shows that the functionals as well as the operator \( s_n \) have norms not exceeding \( A_n \). To show that these norms are actually equal to \( A_n \), it is sufficient to find, for a given \( x \in K \) and \( \epsilon > 0 \) a continuous function \( g \) for which \( \|g\| = 1 \),

\[
s_n(g, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t)D_n(t-x) \, dt > A_n - \epsilon. \quad (2.20)
\]

16
Actually such $g$ can be found in the following way. For the function $g_0(t) = \text{sign } D_n(t-x)$, we have $|D_n(t-x)| = g_0(t)D_n(t-x)$ and hence

$$
\frac{1}{\pi} \int_{-\pi}^{\pi} g_0(t)D_n(t-x)dt = A_n.
$$

But $g_0$ is not continuous, it has jump discontinuities at the finitely many points $t_\gamma$ of $[\pi, \pi]$ where $D_n(t-x)$ changes sign. We surround each $t_\gamma$ by a small interval $I_\gamma = (t_\gamma - \delta, t_\gamma + \delta)$ so as to obtain a continuous function $g$, which has values between $-1$ and $1$ everywhere and coincides with $g_0$ outside $I_\gamma$. That is, if we let $\bigcup I_\gamma = E$, then we have $g = g_0$ on $[-\pi, \pi] \setminus E$.

Consider the difference between the integrals:

$$
\frac{1}{\pi} \left| \int_{-\pi}^{\pi} gD_n dt - \int_{-\pi}^{\pi} g_0D_n dt \right| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} gD_n dt + \int_{E} gD_n dt - \int_{[-\pi,\pi] \setminus E} g_0D_n dt - \int_{E} g_0D_n dt \right|
$$

$$
= \frac{1}{\pi} \left| \int_{E} gD_n dt - \int_{E} g_0D_n dt \right|
$$

$$
= \frac{1}{\pi} \left| \int_{E} (g - g_0)D_n dt \right| \leq 2 \int_{E} |D_n| dt = \mu(E).M < \epsilon
$$
for sufficiently small $\delta$. So we have

$$-\epsilon < \frac{1}{\pi} \int_{-\pi,\pi} g D_n dt - A_n < \epsilon.$$  

(2.21)

Let us write

$$\frac{1}{\pi} \int_{-\pi,\pi \setminus E} g_0 D_n dt = \frac{1}{\pi} \int_{-\pi,\pi} g_0 D_n dt - \frac{1}{\pi} \int_{E} g_0 D_n dt.$$  

By using the first part of the inequality (2.21), we have

$$\| s_n \| = \sup \frac{|s_n(g, x)|}{\| g \|} = s_n(g, x) = \frac{1}{\pi} \int_{-\pi,\pi} g D_n dt > A_n - \epsilon.$$  

(2.22)

Hence by (2.19) and (2.22), we obtain $\| s_n \| = A_n$.

To obtain an asymptotic formula for $A_n$, we write $A_n = \pi^{-1} \int_{0}^{2\pi} 2|D_n(t)| dt$, since $D_n(t)$ (linear combination of cosines) is even. So the function under the integral sign is equal to:

$$2|D_n(t)| = \left| \frac{2 \sin(2n + 1)(t/2)}{2 \sin(t/2)} \right| = \frac{|\sin nt \cdot \cos(t/2) + \sin(t/2) \cdot \cos nt|}{|\sin(t/2)|} = |2/t \sin nt + (\cot t/2 - 2/t) \sin nt + \cos nt|.$$  

It is known that $\cot t/2 - 2/t$ is bounded in $(0, \pi)$.

Hence;

$$A_n = \frac{2}{\pi} \int_{0}^{\pi} \frac{|\sin nt|}{t} dt + O(1).$$  

(2.23)

The integral of $t^{-1} |\sin nt|$ over $(0, \pi/n)$ is bounded, since $|\sin nt| < nt$. We can divide the interval into $n$ equal parts to have the following sum:

$$A_n = \frac{2}{\pi} \left[ \int_{0}^{2\pi/n} \frac{|\sin nt|}{t} dt + \int_{2\pi/n}^{3\pi/n} \frac{|\sin nt|}{t} dt + \cdots + \int_{(n-1)\pi/n}^{\pi} \frac{|\sin nt|}{t} dt \right] + O(1)$$

$$= \frac{2}{\pi} \sum_{k=1}^{n-1} \int_{k\pi/n}^{(k+1)\pi/n} \frac{|\sin nt|}{t} dt + O(1).$$  

(2.24)

Note that the value of the integral over $(0, \pi/n)$ is added to $O(1)$. Now let us make a substitution

$$t = t_k + \frac{k\pi}{n}.$$  

18
If \( t = \frac{k\pi}{n} \), and \( t = \frac{(k+1)\pi}{n} \), we have \( t_k = 0 \), and \( t_k = \frac{\pi}{n} \), respectively. So the equation (2.24) becomes:

\[
A_n = \frac{2}{\pi} \int_{0}^{\pi/n} \sin nt \sum_{k=1}^{n-1} \frac{1}{t + \frac{k\pi}{n}} dt + O(1).
\]

Let

\[
S(t) = \sum_{k=1}^{n-1} \frac{1}{t + \frac{k\pi}{n}} = \frac{1}{t + \frac{\pi}{n}} + \frac{1}{t + \frac{2\pi}{n}} + \cdots + \frac{1}{t + \frac{(n-1)\pi}{n}}.
\]

For \( 0 \leq t \leq \pi/n \), \( S(t) \) lies between \( S(0) \) and \( S(\pi/n) \) where

\[
S(0) = \frac{1}{\pi} + \frac{1}{2\pi} + \cdots + \frac{1}{(n-1)\pi} = \frac{n}{\pi} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right],
\]

and

\[
S(\pi/n) = \frac{n}{\pi} \left[ \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right] = \frac{n}{\pi} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \right] - \frac{n}{\pi} + 1.
\]

\[
\lim_{n \to \infty} \frac{-n + 1}{n} = \frac{-1}{\pi} \Rightarrow \frac{-n}{\pi} + \frac{1}{n} = O(n).
\]

So we have

\[
S(\pi/n) = S(0) + O(n).
\]

It is well-known that the sequence \((D_n)_{n \in \mathbb{N}}\) where

\[
D_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln n,
\]

is convergent and its limit, usually denoted by \( \gamma \), is called Euler constant. For the estimates you may see [6].

So we will write

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = \log n + O(1).
\]

Also we have

\[
\int_{0}^{\pi/n} \sin nt dt = \frac{1}{n} \left[ \cos 0 - \cos \pi \right] = \frac{2}{n}.
\]

Then,

\[
S(0) = \frac{n}{\pi} \left[ \log n + O(1) \right] = \frac{n}{\pi} \log n + O(n),
\]

\[
S(\pi/n) = \frac{n}{\pi} \left[ \log n + O(1) \right] + O(n) = \frac{n}{\pi} \log n + O(n).
\]
Thus the equation (2.25) becomes

$$\|s_n\| = A_n = \frac{2}{\pi} \int_0^{\pi/n} \sin nt \sum_{k=1}^{n-1} \frac{1}{t + \frac{k\pi}{n}} dt + O(1)$$

$$= \frac{2}{\pi n} \left[ \frac{n}{\pi} \left[ \log n + O(1) \right] + O(n) \right] + O(1)$$

$$= \frac{4}{\pi^2} \log n + \frac{4}{\pi n} O(n) + O(1)$$

$$= \frac{4}{\pi^2} \log n + O(1).$$

We see that $\|s_n\|$ is not bounded. ■
3 Approximation Theorems

In this section we will be interested in the uniform convergence of positive linear operators.

Let $A$ be a compact space. Let $C[A]$ be a space of all continuous real-valued functions defined on $A$ with the supremum norm. Let $(L_n)_{n \geq 1}$ be a sequence of positive linear operators defined on $C[A]$. We say that $L_n(f)_{n \geq 1}$ converges to $f$ uniformly on $A$, if

$$\forall \epsilon > 0 \exists N(\epsilon) \in \mathbb{N} \text{ such that } \|L_n(f,x) - f(x)\| < \epsilon \text{ when } n > N, \forall x \in A, \forall f \in C[A].$$

Let $f_1, f_2, \cdots, f_m$ be continuous real functions defined on $A$ that have the following property:

Property *: There exists continuous real functions $a_i(y), y \in A, i = 1, 2, \cdots, m$ such that

$$P_y(x) = \sum_{i=1}^{m} a_i(y)f_i(x)$$

is positive, and equal to zero if and only if $x = y$.

**Theorem 3.1** If the functions $f_1, f_2, \cdots, f_m$ satisfy the Property * and if $L_n$ is a sequence of positive linear operators that map $C[A]$ into itself and satisfy

$$L_n(f_i, x) \rightarrow f_i(x) \text{ uniformly for } x \in A, \ i = 1, 2, \cdots, m \quad (3.1)$$

then

$$L_n(f, x) \rightarrow f(x) \text{ uniformly in } x \text{ for each } f \in C[A]. \quad (3.2)$$

**Proof.** We begin with some properties of functions $P(x) = \sum_{i=1}^{m} a_i f_i(x)$. There exists a $\bar{P}$ with $\bar{P}(x) > 0$ for all $x \in A$. Because if $y_1 \neq y_2$ are two points of $A$, we can take
\( \bar{P}(x) = P_{y_1} + P_{y_2} \). If \( P(x) = \sum_{i=1}^{m} a_i f_i(x) \), then \( L_n P(x) = \sum_{i=1}^{m} a_i L_n(f_i, x) \)...
by linearity of \( L_n \).

\[
\|L_n P(x) - P(x)\| = \sup_{x \in A} \left| \sum_{i=1}^{m} a_i [L_n(f_i, x) - f_i(x)] \right| \leq (\max_{\substack{y \in A}} |a_i|) \sum_{i=1}^{m} \|L_n(f_i, x) - f_i(x)\| < M.m.\epsilon
\]

Hence

\[
L_n P(x) \to P(x), \quad \text{as} \quad n \to \infty \tag{3.3}
\]

uniformly in \( x \) for each \( P \) with constant coefficients. We also have the followings:

\[
P_y(y) = \sum_{i=1}^{m} a_i(y) f_i(y) = 0, \quad L_n(P_y, y) = \sum_{i=1}^{m} a_i(y) L_n(f_i, y), \tag{3.4}
\]

Since \( a_i \)'s are bounded on the compact set \( A \), we can write

\[
\|L_n(P_y, y) - P_y(y)\| = \left\| \sum_{i=1}^{m} a_i(y) [L_n(f_i, y) - f_i(y)] \right\|
\]

\[
= \sup_{y \in A} \left| \sum_{i=1}^{m} a_i(y) [L_n(f_i, y) - f_i(y)] \right|
\]

\[
\leq (\max_{y \in A} |a_i(y)|) \sum_{i=1}^{m} \|L_n(f_i, y) - f_i(y)\| < K.m.\epsilon
\]

Hence we have \( L_n(P_y, y) \to P_y(y) = 0 \) if \( \epsilon \) tends to zero.

Finally, since \( \bar{P}(x) > 0 \), we have

\[
1 = e(x) \leq a \bar{P}(x), \quad x \in A, \tag{3.5}
\]

for some \( a > 0 \). Also observe that

\[
\|L_n(e)\| = \frac{L_n(e, x)}{\|e\|} = L_n(e, x).
\]

Now, apply \( L_n \) to both sides of (3.5) to obtain

\[
L_n(e, x) \leq L_n(a \bar{P}, x) = aL_n\bar{P}(x).
\]

By (3.3), we can easily write

\[
L_n(e, x) \leq aL_n\bar{P}(x) \to a\bar{P}(x).
\]

This implies that for some constant \( M_0 > 0 \),

\[
L_n(e, x) = \|L_n(e)\| \leq M_0. \tag{3.6}
\]

In order to complete the proof, we have to consider the next lemma:
Lemma 3.1 Let \( f_y \in C[A] \), \( y \in A \), be a family of functions for which \( f_y(x) \) is a continuous function of the point \( (x,y) \in A \times A \) and \( f_y(y) = 0 \) for all \( y \in A \). Then \( L_n(f_y,y) \to 0 \) uniformly in \( y \).

Proof. Consider the diagonal set \( B = \{(y,y)\} \) in \( A \times A \) and some \( \epsilon > 0 \). Since \( f_y \) is continuous, we have \( |f_y(x) - f_y(y)| = |f_y(x)| < \epsilon \) when \( (x,y) \in U \). Note that the union \( G \) of all these \( U \) is an open set. Note also that its complement \( F \) is compact since it is a closed subset of a compact set \( A \times A \). Now let us denote

\[
m = \min_{(x,y) \in F} P_y(x) > 0, \quad M = \max_{(x,y) \in F} |f_y(x)|.
\]

In fact, we have \( |f_y(x)| < \epsilon \) if \( (x,y) \in G \), and \( |f_y(x)| < \frac{M}{m} P_y(x) \) if \( (x,y) \in F \). Therefore we can write

\[
|f_y(x)| < \epsilon + \frac{M}{m} P_y(x).
\]

In (3.7) let \( x = y \). So we have

\[
|f_y(y)| < \epsilon e(y) + \frac{M}{m} P_y(y).
\]

The positivity of \( L_n \) and (3.6) implies that

\[
|L_n(f_y,y)| < \epsilon L_n(e,y) + \frac{M}{m} L_n(P_y,y) \leq \epsilon(M_0 + 1).
\]

Hence for all large \( n \), we have \( L_n(f_y,y) \to 0 \) uniformly in \( y \).

Now we can complete the proof of the Theorem (3.1) easily. If \( f \in C[A] \) is given, we put:

\[
f_y(x) = f(x) - \frac{f(y)}{P(y)} \bar{P}(x).
\]

(3.8)

Let \( x = y \) and apply \( L_n \) to both sides of the equation (3.8) to obtain

\[
L_n(f_y,y) = L_n(f,y) - \frac{f(y)}{P(y)} L_n(\bar{P},y) \to 0.
\]

By (3.3) we have \( L_n(\bar{P},y) \to \bar{P}(y) \). Hence \( L_n(f,x) \to f(x) \) uniformly in \( x \) for each \( f \in C[A] \).

Let us recall Theorem of Korovkin once more.
Theorem 3.2 (Theorem of Korovkin) Let \( L_n : C[a,b] \to C[a,b] \) be a sequence of positive linear operators, and let \( p_k(x) = x^k \). If \( L_n(p_k) \) converges uniformly to \( p_k \) on \( [a,b] \) for \( k = 0, 1, 2 \), then the sequence \( \{L_n(f)\} \) converges uniformly to \( f \) on \( [a,b] \) for each \( f \) in \( C[a,b] \).

Proof. Let \( p_0 = f_1 = 1, p_1 = f_2 = x, p_2 = f_3 = x^2 \). If we take \( a_1(y) = y^2, a_2(y) = -2y, \) and \( a_3(y) = 1 \), we have \( P_y(x) = y^2 f_1 - 2y f_2 + f_3 = y^2 - 2yx + x^2 = (y-x)^2 > 0 \), when \( x \neq y \) and \( P_y(x) = 0 \) if and only if \( x = y \) which means that the Property* is satisfied.

Since by assumption \( L_n(f_k)(x) \to f_k(x) = x^k \) uniformly for \( k = 1, 2, 3 \), then we have \( L_n(f) \to f \) for arbitrary \( f \in C[a,b] \), by theorem (3.1) \( \blacksquare \)

Remark 3.1 Theorem (3.1) allows us to check the convergence of certain operators with a minimum of calculations.

Now let us consider Weiestrass approximation theorem:

Theorem 3.3 (Weiestrass Approximation Theorem) Each continuous real function \( f \) on \( [a,b] \) is approximable by algebraic polynomials: For each \( \epsilon > 0 \) there is a polynomial \( P_n(x) = \sum_{k=0}^{n} a_k x^k \) with
\[
|f(x) - P_n(x)| < \epsilon, \quad a \leq x \leq b.
\]

The linear substitution \( t = (x-a)/(b-a) \) reduces the interval \( a \leq x \leq b \) to the interval \( 0 \leq t \leq 1 \). Thus Theorem (3.3) follows from:

Theorem 3.4 If the function \( f \) is continuous on \([0,1]\), then
\[
\lim_{n \to \infty} B_n(f,x) = f(x) \quad \text{uniformly for} \quad 0 \leq x \leq 1.
\]

Proof. For the polynomials \( p_{nk} \) of (2.9), we have
\[
\sum_{k=0}^{n} kp_{nk}(x) = \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k}
= \sum_{k=0}^{n} \frac{n(n-1)!}{(n-k)!k!} x^{k-1}(1-x)^{n-k}
= nx \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^{k-1}(1-x)^{n-k}
= nx \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} x^k (1-x)^{n-1-k} = nx
\]
\[
\sum_{k=0}^{n} k(k-1)p_{nk}(x) = \sum_{k=0}^{n} k(k-1)\binom{n}{k}x^k(1-x)^{n-k} \\
= \sum_{k=0}^{n} k(k-1)\frac{n(n-1)(n-2)!}{(n-k)!(k-1)!(k-2)!}x^k(1-x)^{n-k} \\
= n(n-1)x^2\sum_{k=0}^{n} \frac{(n-2)!}{(n-k)!(k-2)!}x^{k-2}(1-x)^{n-k} \tag{3.10} \\
= n(n-1)x^2\sum_{k=0}^{n-2} \frac{(n-2)!}{(n-2-k)!k!}x^k(1-x)^{n-2-k} \\
= n(n-1)x^2
\]

so that
\[
\sum_{k=0}^{n} k^2p_{nk}(x) = n(n-1)x^2 - nx = n^2x^2 + nx(1-x). \tag{3.11}
\]

Now consider the functions \(f_1 = 1, f_2 = x, f_3 = x^2\). We observed in the proof of the Theorem (3.2) that they satisfy the Property*. Here we have Bernstein polynomials as positive linear operators. Recall that

\[
B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).
\]

By (2.9), (3.9), and (3.11), we have

\[
\lim_{n \to \infty} B_n(1, x) = 1. \\
\lim_{n \to \infty} B_n(x, x) = \lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k}x^k(1-x)^{n-k}\frac{k}{n} = \lim_{n \to \infty} \frac{nx}{n} = x. \\
\lim_{n \to \infty} B_n(x^2, x) = \lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k}x^k(1-x)^{n-k}\frac{k^2}{n^2} \\
= \lim_{n \to \infty} \frac{n^2x^2 + nx(1-x)}{n^2} = \lim_{n \to \infty} \left[\frac{x^2 + n^{-1}x(1-x)}{n}\right] = x^2.
\]

So Theorem (3.1) is satisfied. Then \(B_n(f) \to f\) uniformly for \(0 \leq x \leq 1\).

Now, let us consider Weiestrass Approximation Theorem for \(2\pi\)-periodic functions:

**Theorem 3.5 (Weierstrass)** If \(f \in C^*, \) then for each \(\epsilon > 0\) there exists a trigonometric polynomial \(T\) such that

\[|f(x) - T(x)| < \epsilon, \quad x \in K.\]
Proof. We will prove this theorem just as in [2]. We can take \( f_1 = 1, f_2 = \cos x, \) and \( f_3 = \sin x, \) if we take \( a_1(y) = 1, a_2(y) = -\cos y, \) and \( a_3(y) = \sin y, \) we have
\[
P_y(x) = 1 - \cos y \cos x - \sin y \sin x = 1 - \cos(y - x) > 0, \quad \text{when } x \neq y \quad \text{and } P_y(x) = 0 \text{ if and only if } x = y,
\]
which means that the Property* is again satisfied. We can show that \( \sigma_n(f) \to f \) for all \( C^* = C[K]. \) Thus, Theorem (3.5) follows from Theorem (3.1).
4 Variations on a Theorem of Korovkin

Now, in this section, we will mention what have been done in the article [3]. Let us begin with the notations used in the article. By $C(X)$ we mean the Banach space of real-valued continuous functions over $X$ with the supremum norm $\|\cdot\|_\infty$, that is, $\|f\|_\infty := \sup_{x \in X}|f(x)|$ for $f$ in $C(X)$.

We have seen in Example (2.4) and Example (2.5) that choosing the supremum norm on $C(X)$ is meaningful.

**Definition 4.1** For $\alpha$ in $C(X)$, $Z(\alpha)$ signifies the set of zeros of $\alpha$, i.e.,

$$Z(\alpha) = \{x \in X : \alpha(x) = 0\}.$$

All the approximations that will be discussed in this article depend basically on the next two lemmas:

**Lemma 4.1** Let $X$ be a compact metric space, and let $\alpha$ and $\beta$ be positive functions from $C(X)$. Suppose that $Z(\alpha) \subset Z(\beta)$. Then for each $\epsilon > 0$ there exists $M = M(\epsilon) > 0$ such that

$$\alpha(x) \leq \epsilon + M\beta(x)$$

holds for each $x$ in $X$.

**Proof.** Let $X = Z(\beta) \cup [Z(\alpha) \setminus Z(\beta)] \cup [X \setminus Z(\alpha)]$. 

---

27
Case 1. Let $x \in Z(\beta)$. Then $\beta(x) = 0$, $\alpha(x) = 0$. Hence $\alpha(x) = \beta(x)$ in $Z(\beta)$.

Case 2. Let $x \in Z(\alpha) \setminus Z(\beta)$. Then we have $x \in Z(\alpha)$ and $x \notin Z(\beta)$. So $\alpha = 0$ and $\beta \neq 0$. Hence $\beta > 0$. Therefore $\alpha(x) < \beta(x)$ in $Z(\alpha) \setminus Z(\beta)$.

Case 3. Let $x \in X \setminus Z(\alpha)$. Let us consider the difference of two continuous functions $\alpha$ and $\beta$. Note that in that set we have $\alpha > 0$, $\beta > 0$ and the following:

Given any $\epsilon > 0$ $\exists \delta_1(\epsilon)$ such that $|\alpha(x) - \alpha(x_0)| < \epsilon$ and $\exists \delta_2(\epsilon)$ such that $|\beta(x) - \beta(x_0)| < \epsilon$ where $x_0 \in X \setminus Z(\alpha)$. So

$$|\alpha(x) - \beta(x)| = |\alpha(x) - \alpha(x_0) + \alpha(x_0) - \beta(x_0) + \beta(x_0) + \beta(x)|$$

$$\leq |\alpha(x) - \alpha(x_0)| + |\alpha(x_0) - \beta(x_0)| + |\beta(x_0) + \beta(x)|$$

Since $\alpha, \beta$ are positive, there exist $M_1 > 0$ such that $|\alpha(x_0) - \beta(x_0)| = M_1 \beta(x_0) = K$. So for arbitrary $\epsilon_1$ if we choose $\delta = \min\{\delta_1, \delta_2\}$ we have $|\alpha(x) - \beta(x)| \leq 2\epsilon_1 + K$ or $\epsilon - K \leq \alpha(x) - \beta(x) \leq \epsilon + K$. Using the second part of the inequality, we get $\alpha(x) \leq \epsilon + K + \beta(x)$ or simply $\alpha(x) \leq \epsilon + M\beta(x)$, $\forall x \in X$, where $K + \beta(x) = M\beta(x)$ for some constant $M > 0$. ■

Now, we will give two definitions which are relevant to our subsequent discussions.

**Definition 4.2** Let $f \in C(X)$. The **diagonal** $\Delta(f)$ of $f$ in $X$ is defined by

$$\Delta(f) = \{(x,t) \in X \times X : f(x) = f(t)\}.$$

**Definition 4.3** Let $f$ belong to $C(X)$. If $\gamma$ is a positive function in $C(X \times X)$ such that

$$Z(\gamma) \subset \Delta(f),$$

then $\gamma$ is a **bounding function** for $f$.

**Example 4.1** Let $f(x) = x^2$. Then $\gamma(x,t) = (x - t)^2$ is a bounding function of $f$.

Assume $f(x) = f(t)$. Then $x^2 = t^2$. Hence $\Delta(f) = \{(x,t) : x = t$ or $x = -t\}$. On the other hand, if $\gamma(x,t) = (x - t)^2 = 0$, then $Z(\gamma) = \{(x,t) : x = t\}$. Therefore we have $Z(\gamma) \subset \Delta(f)$. 

28
Example 4.2 If \( g(x) = 1 - \cos x \), then \( \gamma(x, t) = 1 - \cos(x - t) \) is a bounding function of \( g \).

Assume \( g(x) = g(t) \), then \( \Delta(g) = \{(x, t) : x = t + 2k\pi \text{ or } x = -t + 2k\pi, k \in \mathbb{Z}\} \).

On the other hand, if \( \gamma(x, t) = 1 - \cos(x - t) = 0 \), then \( \cos(x - t) = 1 \). Hence \( Z(\gamma) = \{(x, t) : x = t + 2k\pi, k \in \mathbb{Z}\} \). Therefore \( Z(\gamma) \subset \Delta(f) \).

Let \( \gamma(x, t) = 1 - \cos(x - t) = 0 \). Then \( \cos(x - t) = 1 \). Hence \( Z(\gamma) = \{(x, t) : x = t + 2k\pi, k \in \mathbb{Z}\} \). Therefore \( Z(\gamma) \subset \Delta(f) \).

Let \( \gamma \) in \( C(X \times X) \) be positive. For each \( t \) in \( X \), we write \( \gamma_t(x) = \gamma(x - t) \). Given an operator \( L : C(X) \to C(X) \), the function \( L(\gamma_t) \) is the result of applying the operator to \( \gamma \) regarded as a function of the first variable, \( x \), only. With this notation, now we are ready to give the next lemma:

Lemma 4.2 Let \( X \) be a compact metric space, let \( f \) belong to \( C(X) \), let \( \gamma \) be a bounding function for \( f \), and let \( L_n : C(X) \to C(X) \) be a sequence of positive operators. Suppose that

(a) \( L_n(1) \) converges uniformly to 1 on \( X \) and

(b) \( L_n(\gamma_t)(t) \) converges uniformly in \( t \) to 0 on \( X \).

Then \( L_n(f) \) converges uniformly to \( f \) on \( X \).

Proof. Let \( \alpha \in C(X \times X) \) be given by \( \alpha(x, t) = |f(x) - f(t)| \). Notice that \( Z(\alpha) = \Delta(f) \).

(i) \( Z(\alpha) \subseteq \Delta(f) \)

If \( (x, t) \in Z(\alpha) \), then \( \alpha(x, t) = |f(x) - f(t)| = 0 \), and hence \( f(x) = f(t) \). So \( (x, t) \in \Delta(f) \).

(ii) \( \Delta(f) \subseteq Z(\alpha) \)

\( (x, t) \in \Delta(f) \) implies \( f(x) = f(t) \). Then \( |f(x) - f(t)| = \alpha(x, t) = 0 \). So we have \( (x, t) \in Z(\alpha) \).

Let \( \epsilon > 0 \). Since \( \gamma \) is a bounding function for \( f \), we have \( Z(\gamma) \subseteq \Delta(f) = Z(\alpha) \).

By invoking Lemma (4.1), we can find \( M > 0 \) such that

\[
\alpha(x, t) = |f(x) - f(t)| \leq \epsilon + M\gamma(x, t),
\]

for all \( (x, t) \in X \times X \). Now, for fixed \( t \) in \( X \), the positivity of \( L_n \) implies that

\[
L_n|f(x) - f(t)| \leq L_n(\epsilon + M\gamma(x, t)),
\]

29
\[ |L_n f(x) - f(t) L_n(1)(x)| \leq \epsilon L_n(1)(x) + M L_n(\gamma t)(x), \]

for each \( x \) in \( X \times X \). In particular, if \( x = t \), then

\[
|L_n(f)(t) - f(t)| = |L_n(f)(t) - f(t) L_n(1)(t) + f(t) L_n(1)(t) + f(t)| \\
\leq |L_n(f)(t) - f(t) L_n(1)(t)| + |f(t) L_n(1)(t) + f(t)| \\
\leq \epsilon L_n(1)(t) + M L_n(\gamma t)(t) + |f(t)| |L_n(1)(t) - 1| \\
\leq \epsilon L_n(1)(t) + M L_n(\gamma t)(t) + |f(t)| \infty |L_n(1)(t) - 1|
\]

For small \( \epsilon \) we have \( \epsilon L_n(1)(t) \to 0 \). By the assumptions given in the Lemma (4.2) we also have

\( M L_n(\gamma t)(t) \to 0, \quad |f(t)| \infty |L_n(1)(t) - 1| \to 0. \)

Therefore \( L_n(f)(t) \to f(t) \), uniformly in \( t \), for sufficiently large \( n \). This shows that \( L_n(f) \) converges uniformly to \( f \). \( \blacksquare \)

Before going further, let us define the multiplication of two functions in the following way:

\[
(fg)(x) = f(x)g(x), \quad \forall x \in X.
\]

Notice that for \( f, g \in C(X) \) and for real \( a \) diagonal satisfies the following properties.

**Properties**:

(i) \( \Delta(f) \cap \Delta(g) \subset \Delta(f + g) \)

(ii) \( \Delta(f) \cap \Delta(g) \subset \Delta(fg) \)

(iii) \( \Delta(f) \subset \Delta(af) \)

**Proof.** Let \( (x, t) \in \Delta(f) \cap \Delta(g) \). Then we have \( f(x) = f(t), \quad g(x) = g(t) \). Hence we can write:

(i) \( f(x) + g(x) = f(t) + g(t) \) imply \( (f + g)(x) = (f + g)(t) \), then \( (x, t) \in \Delta(f + g) \).

(ii) \( f(x)g(x) = f(t)g(t) \) imply \( (fg)(x) = (fg)(t) \), then \( (x, t) \in \Delta(fg) \).

(iii) \( af(x) = af(t) \) imply \( (af)(x) = (af)(t) \), then \( (x, t) \in \Delta(af) \).

\( \blacksquare \)
Now let $C_\gamma(X)$ be a set of continuous functions for which $\gamma$ is a bounding function, that is

$$f \in C_\gamma(X) \Rightarrow Z(\gamma) \subset \Delta(f). \quad (4.1)$$

Assume that $f, g \in C_\gamma(X)$. So by (4.1) we have

$$Z(\gamma) \subset \Delta(f), \text{ and } Z(\gamma) \subset \Delta(g) \Rightarrow Z(\gamma) \subset \Delta(f) \cap \Delta(g).$$

Then, by the properties $(i)-(iii)$ observed above, we have

1. $Z(\gamma) \subset \Delta(f + g)$ imply $f + g \in C_\gamma(X)$
2. $Z(\gamma) \subset \Delta(af)$ imply $af \in C_\gamma(X)$
3. $Z(\gamma) \subset \Delta(fg)$ imply $fg \in C_\gamma(X)$

(1) and (2) show that $C_\gamma(X)$ is a linear subspace of $C(X)$. Since we also have (3), $C_\gamma(X)$ is, in fact, a subalgebra of $C(X)$.

Thus condition (a) in Lemma (4.2) can be replaced, without loss of generality, by the requirement that $L_n(1) \equiv 1$ for all $n$ (see [3]).

Finally, Korovkin’s theorem follows readily from Lemma (4.2) by taking $X = [a,b]$ and using a bounding function for arbitrary $f$ in $C [a,b]$ the function

$$\gamma(x,t) = (x-t)^2.$$ 

Observe that $L_n(1) \to 1$ uniformly is satisfied automatically from the assumptions given in the theorem. Also we have

$$L_n(\gamma_t)(x) = L_n(x-t)^2 = L_n(x^2) - 2tL_n(x) + t^2L_n(1) \to x^2 - 2tx + t^2$$

uniformly as $n$ tends to infinity. Hence $L_n(\gamma_t)(t) \to 0$ uniformly in $t$. So for arbitrary $f \in C [a,b]$, we have $L_n(f) \to f$ uniformly on $[a,b]$ by Lemma (4.2).
5 Applications to Classical Results

Our aim in this section is to observe the way of using the Lemma (4.2). Remember that a different proof for Weiestrass Approximation Theorem was given in [2]. But we will now apply Lemma (4.2) to prove this theorem.

Proof. Here, as a sequence of positive linear operators, we have again Bernstein Polynomials (2.8).

\[ L_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k}{n} \right). \]

Take as \( \gamma \) the function \( \gamma(x, t) = (x - t)^2 \). In this case we have \( L_n(1) \equiv 1 \) by equation (2.9).

\[
\begin{align*}
L_n(\gamma_t)(t) &= \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} \gamma_t \left( \frac{k}{n} \right) \\
&= \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} \left( \frac{k}{n} - t \right)^2 \\
&= \sum_{k=0}^{n} k^2 n^2 \binom{n}{k} t^k (1 - t)^{n-k} - \sum_{k=0}^{n} 2kt \binom{n}{k} t^k (1 - t)^{n-k} + \sum_{k=0}^{n} t^2 \binom{n}{k} t^k (1 - t)^{n-k} \\
&= t^2 + \frac{t(1-t)}{n} - 2t^2 + t^2 = \frac{t(1-t)}{n}
\end{align*}
\]

by equations (2.9), (3.9), and (3.11). Then \( L_n(\gamma_t)(t) \rightarrow 0 \) uniformly as \( n \) tends to infinity. So for arbitrary \( f \in C(X) \), \( L_n(f) \rightarrow f \) uniformly on \( X \) by Lemma (4.2). \( \blacksquare \)

**Theorem 5.1 (Poisson’s Formula).** Let \( f \) in \( C[-\pi, \pi] \) satisfy \( f(-\pi) = f(\pi) \). If \( u(r, \theta) \) is defined on \( [0, 1) \times [-\pi, \pi] \) by

\[
u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) f(s)}{1 - 2r \cos(\theta - s) + r^2} \, ds,
\]

then \( \lim_{r \to 1^-} u(r, \theta) = f(\theta) \) for each \( \theta \) in \( [-\pi, \pi] \) and the convergence is uniform on \( [-\pi, \pi] \).
Proof. Now, we will use the following fact:

\[
\lim_{r \to 1^-} u(r, \theta) = f(\theta)
\]

uniformly on \([-\pi, \pi]\) if and only if

\[
\lim_{n \to \infty} u(r_n, \theta) = f(\theta)
\]

uniformly on \([-\pi, \pi]\) for each sequence \((r_n) \uparrow 1\).

Let \((r_n)\) be a sequence such that \((r_n) \uparrow 1\), and define \(L_n : C[-\pi, \pi] \to C[-\pi, \pi]\) by

\[
L_n(f)(\theta) = u(r_n, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2 f(s))}{1 - 2r_n \cos(\theta - s) + r_n^2} \, ds.
\]

Note that \((1 - r_n^2) \geq 0\) and \(1 - 2r_n \cos(\theta - s) + r_n^2 \geq 1 - 2r_n \cos(\theta - s) + r_n^2 \cos^2(\theta - s) = (1 - r_n \cos(\theta - s))^2 \geq 0\).

So \(L_n\) is positive.

In addition to this,

\[
L_n(f + g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2 (f + g)(s))}{1 - 2r_n \cos(\theta - s) + r_n^2} \, ds
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2 f(s))}{1 - 2r_n \cos(\theta - s) + r_n^2} \, ds + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2 g(s))}{1 - 2r_n \cos(\theta - s) + r_n^2} \, ds
= L_n(f)(\theta) + L_n(g)(\theta).
\]

\[
L_n(\alpha f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2 \alpha f)(s)}{1 - 2r_n \cos(\theta - s) + r_n^2} \, ds = \alpha L_n(f)(\theta).
\]

So the calculations in (5.2) show that \(L_n\) is linear.

Now, use \(\gamma(x, t) = 1 - \cos(x - t)\) as a bounding function. Then

\[
L_n(\gamma_t)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2 \gamma_t(s))}{1 - 2r_n \cos(t - s) + r_n^2} \, ds
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r_n^2)(1 - \cos(s - t))}{1 - 2r_n \cos(t - s) + r_n^2} \, ds.
\]

We claim that:

\[
0 \leq \frac{1 - \cos(s - t)}{1 - 2r_n \cos(t - s) + r_n^2} \leq \frac{2}{(1 + r_n)^2}.
\]

(5.3)

Let us use standard method of calculus. Assume

\[
f(\theta) = \frac{1 - \cos \theta}{1 - 2r_n \cos \theta + r_n^2}.
\]
By taking the first derivative of \( f \) with respect to \( \theta \), we get:

\[
f'(\theta) = \frac{\sin \theta (1 - 2r_n \cos \theta + r_n^2) - (1 - \cos \theta)(2r_n \sin \theta)}{(1 - 2r_n \cos \theta + r_n^2)^2} = \frac{(1 - r_n)^2 \sin \theta}{(1 - 2r_n \cos \theta + r_n^2)^2}.
\]

\( f'(\theta) = 0 \) implies \( \sin(\theta) = 0 \). So for \( \theta = 0, \pi, -\pi \) we have extremals.

\[
f''(\theta) = \frac{(1 - r_n)^2 \cos \theta (1 + r_n^2 - 2r_n \cos \theta)^2}{(1 + r_n^2 - 2r_n \cos \theta)^4} - \frac{(1 - r_n)^2 \sin \theta [2(1 + r_n^2 - 2r_n \cos \theta)(2r_n \sin \theta)]}{(1 + r_n^2 - 2r_n \cos \theta)^4}
= \frac{(1 - r_n)^2 \cos \theta}{(1 + r_n^2 - 2r_n \cos \theta)^2} - \frac{(1 - r_n)^2 4r_n \sin^2 \theta}{(1 + r_n^2 - 2r_n \cos \theta)^3}.
\]

\[
f''(0) = \frac{1}{(1 - r_n)^2} > 0
f''(\pi) = f''(-\pi) = -\frac{(1 - r_n)^2}{(1 + r_n)^4} < 0
\]

To obtain the maximum value of \( f \), we let \( \theta = \pi \). So we have:

\[
0 \leq \frac{1 - \cos(s - t)}{1 - 2r_n \cos(t - s) + r_n^2} \leq \frac{1 - \cos \pi}{1 - 2r_n \cos \pi + r_n^2} = \frac{2}{(1 + r_n)^2}
\]

Hence

\[
0 \leq L_n(\gamma_t(t)) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2(1 - r_n)^2}{(1 + r_n)^2} ds = \frac{2(1 - r_n)}{1 + r_n} \to 0
\]

Therefore \( L_n(\gamma_t(t)) \) tends to zero uniformly as \( n \) increases.\( \blacksquare \)

Note that (5.1) is the solution of the Dirichlet Problem for the unit disc:

\[
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_\theta = 0, \quad 0 \leq r < 1, \quad u(1, \theta) = h(\theta).
\]

For the proof you may see [7].

**Theorem 5.2 (Fejer Approximation Theorem)** Let \( \Lambda = \{2k\pi : k \in \mathbb{Z}\} \).

For \( n = 1, 2, \cdots \) the Fejer kernel \( K_n \) is defined on \( \mathbb{R} \) by

\[
K_n(t) = \begin{cases} 
\frac{1}{2n} \left( \frac{\sin\left[\frac{1}{2n}t\right]}{\sin\left(\frac{1}{2}\right)} \right)^2 & \text{if } t \notin \Lambda \\
\frac{n}{2} & \text{if } t \in \Lambda
\end{cases}
\]

The functions \( K_n \) have the following property:

If \( f \) in \( C(\mathbb{R}) \) has period \( 2\pi \), then the sequence \( (f_n) \) defined by

\[
f_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) K_n(x - s) ds \quad (5.4)
\]

converges uniformly to \( f \) on \( \mathbb{R} \).
Proof. Notice that here \( f_n \) is, in fact, the arithmetic mean of the \( s_n \) which we have already calculated as (2.16). So \((f_n)\) is a sequence of positive linear operators. Then let us write \( L_n(f)(x) = f_n \). By the equation (2.17), we have \( \|f_n(1)\| = f_n(1) = 1 \). Let us take \( \gamma(x,t) = 1 - \cos(x - t) = 2 \sin^2 [(x - t)/2] \) as a bounding function. Then

\[
L_n(\gamma(t))(t) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} 2 \sin^2 \left[ \frac{(t - s)}{2} \right] \left[ \frac{\sin \frac{n(t-s)}{2}}{\sin \frac{(t-s)}{2}} \right]^2 ds
\]

where \( \xi \in [-\pi, \pi] \). Here we used the Mean Value Theorem for Integrals.

Hence \( L_n(\gamma(t))(t) \to 0 \) uniformly as \( n \) tends to infinity. So Theorem (5.2) follows from Lemma (4.2).

Note that the expressions \( f_n \) (or \( \sigma_n \)) are Cesaro averages used in the theory of Fourier Series. One can find the detailed definitions and applications in [8].

Fejer’s Approximation theorem asserts that the Cesaro averages of the Fourier partial sums of a continous function \( f \) of period \( 2\pi \), converge uniformly to \( f \).

As a consequence of Fejer’s result, we obtain the following:

**Corollary 5.1 (Weiestrass Trigonometric Approximation Theorem)**. If \( f \) in \( C[ -\pi, \pi] \) satisfies \( f(-\pi) = f(\pi) \), then there exists a sequence \( (f_n) \) of trigonometric polynomials that converges uniformly to \( f \) on \( [-\pi, \pi] \).
References


